

# Polynomial Functions

## A Few Rules of Differentiation

So far in this class, we have study differentiation in the context of quadratic functions, cubic functions, and positive power functions. We have not done much in the way of general principles of differentiation. In today's lecture, we begin to study rules which apply to all functions which have derivatives.

The first rule is the **sum rule**. Suppose that  $f(x)$  and  $g(x)$  are two functions with derivatives. We can define a new function,  $h(x)$  which is the sum of these two functions:

$$h(x) = f(x) + g(x).$$

For example, we could have  $f(x) = x^2 + 3x + 4$  and  $g(x) = -2x + 6$ . Then

$$h(x) = f(x) + g(x) = (x^2 + 3x + 4) + (-2x + 6) = x^2 + x + 10.$$

Now suppose that  $f(x)$  and  $g(x)$  have derivatives (in the case of our example, they do). Then the derivative of  $h(x)$  is equal to the sum of the derivatives of  $f(x)$  and  $g(x)$ :

$$\frac{dh}{dx}(x) = \frac{df}{dx}(x) + \frac{dg}{dx}(x).$$

So, to continue our example, let us verify the sum rule: the derivative of  $f(x)$  is  $2x + 3$ , the derivative of  $g(x)$  is  $-2$ , and the derivative of  $h(x)$  is  $2x + 1$ . The sum rule does work in this example:

$$\begin{aligned}\frac{dh}{dx}(x) &= \frac{df}{dx}(x) + \frac{dg}{dx}(x) \\ 2x + 1 &= (2x + 3) + (-2) \\ 2x + 1 &= 2x + 1.\end{aligned}$$

The second rule we are studying today is the **difference rule**. The difference rule states that if  $f(x)$  and  $g(x)$  are two functions with derivatives, then the difference of those two functions,

$$k(x) = f(x) - g(x),$$

also has a derivative and that derivative is equal to the difference between the derivative of  $f(x)$  and the derivative of  $g(x)$ :

$$\frac{dk}{dx}(x) = \frac{df}{dx}(x) - \frac{dg}{dx}(x).$$

Let us use the example above to demonstrate the difference rule: in this case,

$$k(x) = f(x) - g(x) = (x^2 + 3x + 4) - (-2x + 6) = x^2 + 5x - 2,$$

so the derivative of  $k(x)$  is equal to  $2x + 5$ . So, as shown below, the difference rule works in this case:

$$\begin{aligned}\frac{dk}{dx}(x) &= \frac{df}{dx}(x) - \frac{dg}{dx}(x) \\ 2x + 5 &= (2x + 3) - (-2) \\ 2x + 5 &= 2x + 5.\end{aligned}$$

We have one more rule today: the **constant multiple rule**. The constant multiple rule states that if  $f(x)$  is a function with a derivative, and  $c$  is a real number, then  $c$  times  $f(x)$  is also a function with a derivative,

$$m(x) = cf(x),$$

and the derivative of  $m(x)$  is equal to  $c$  times the derivative of  $f(x)$ :

$$\frac{dm}{dx}(x) = c \frac{df}{dx}.$$

So, to use our example above yet again, let  $c = 7$ . Then

$$m(x) = cf(x) = 7 \cdot (x^2 + 3x + 4) = 7x^2 + 21x + 28,$$

so the derivative of  $m(x)$  is  $14x + 21$ . We thus verify the constant multiple rule in this case:

$$\begin{aligned}\frac{dm}{dx}(x) &= c \frac{df}{dx} \\ 14x + 21 &= 7 \cdot (2x + 3) \\ 14x + 21 &= 14x + 21.\end{aligned}$$

## Differentiating Polynomials

Now, at this point in the lecture, you are probably wondering what good it is to know these three rules. The answer is that now you know how to find the derivative of any polynomial.

A polynomial is any function of the form  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ . Do not be too concerned about all of the subscripts and superscripts: all we are trying to write here is that  $f(x)$  consists of an  $x^n$  term for some natural number  $n$ , followed by an  $x^{n-1}$  term, followed by many other terms, under we get to an  $x$  term and a constant term. Writing  $a$  followed by subscripts for the coefficients of these terms is just a way to write polynomials down when we may have more terms than we have letters in the alphabet.

What is important for our purposes is that a polynomial is the sum and difference of many terms which have derivatives, so we can apply the sum and difference rules to find its derivative. So, to take an example, let  $f(x) = 6x^5 + 3x^4 - 9x + 7$ . Applying the sum rule in this case, we get that

$$\frac{df}{dx} = \frac{d(6x^5)}{dx} + \frac{d(3x^4)}{dx} - \frac{d(9x)}{dx} + \frac{d(7)}{dx}.$$

Now, right away, we know how to take the derivatives of the last two terms: the derivative of the linear term is 9, and the derivative of the constant term is 0. So we know that

$$\frac{df}{dx} = \frac{d(6x^5)}{dx} + \frac{d(3x^4)}{dx} - 9 + 0.$$

What about the other terms? We recognize that they are constant multiples of the positive power functions, so we can apply the constant multiple rule to them to find their derivatives. The rest of the calculations are below:

$$\begin{aligned}\frac{df}{dx} &= \frac{d(6x^5)}{dx} + \frac{d(3x^4)}{dx} - 9 + 0 \\ \frac{df}{dx} &= 6 \frac{d(x^5)}{dx} + 3 \frac{d(x^4)}{dx} - 9 \\ \frac{df}{dx} &= 6 \cdot 5x^4 + 3 \cdot 4x^3 - 9 \\ \frac{df}{dx} &= 30x^4 + 12x^3 - 9.\end{aligned}$$

Let us do another example, this time without showing some of the steps explicitly. Let  $g(x) = -2x^7 + 9x^5 - x^2 + 5x$ . Then the derivative of  $g(x)$  is

$$\begin{aligned}\frac{dg}{dx} &= -2 \cdot 7x^6 + 9 \cdot 5x^4 - 2x + 5 \\ \frac{dg}{dx} &= -14x^6 + 45x^4 - 2x + 5.\end{aligned}$$

Notice how we did not actually break the polynomial up into its terms and differentiate, at least on paper. Instead, we simply recognized that this function is a polynomial, that it consists of sums and difference of terms which are constants times positive power functions, so to take the derivative of the polynomial, we

took the derivatives of the positive power functions, multiplied those derivatives by the coefficients of the terms, and then added them up. As another example, take  $h(x) = x^{17} + 2x^{10} - 5x^2 + 4$ :

$$\frac{dh}{dx} = 17x^{16} + 2 \cdot 10x^9 - 5 \cdot 2x$$
$$\frac{dh}{dx} = 17x^{16} + 20x^9 - 10x.$$

It really is that simple.