Name: SOLUTIONS
Section (circle one): Esselstein Setyadi

FINAL EXAM

December 7, 2004

This is a closed book, closed notes exam. You are on your honor not to use outside sources while you are taking this exam. You are also on your honor not to talk about this exam with another student until both you and the other student have handed in your exams. Show all of your work, and clearly indicate your answer.

Problem	Points Possible	Points Earned
1	17	
2	18	
3	16	
4	22	
5	8	
6	56	
7	6	
8	12	
9	18	
10	27	
Total	200	

1. (17 points) Hagrid is raising a litter of Blast-ended Skrewts that reproduce according to the function

$$M(t) = M_0 a^t.$$

He begins with 2 Skrewts. Skrewts are known to reproduce at a rate of 2% a week.

(a) What is the formula, $M(t)$, for the number of Skrewts that Hagrid will have after t weeks?

$M(t) = 2(1.02)^t$

(b) Express $M(t)$ as a function in base e.

$2(1.02) = 2e^k$

$1.02 = e^k$

$\ln(1.02) = k$

$M(t) = 2e^{t\ln(1.02)}$

(c) At what *rate* is the Skrewt population increasing after 10 weeks? Give the units.

$M'(t) = 2\ln(1.02)e^{t\ln(1.02)}$

$M'(10) = 2\ln(1.02)\, e^{10\ln(1.02)}$

2. (18 points) Using shortcuts, find the derivatives of the following functions:

(a) $f(x) = (\arcsin x)(\ln x)$

$f'(x) = \frac{\ln x}{\sqrt{1-x^2}} + \frac{\arcsin x}{x}$

(b) $g(x) = \cos^2(x+1)$

$g'(x) = 2\cos(x+1)(-\sin(x+1))$

$= -2\cos(x+1)\sin(x+1)$

(c) $h(x) = \ln(e^x) + e^{e^{-x}} + x^{2004} + 4002^x$

$h'(x) = 1 - e^{-x}e^{e^{-x}} + 2004x^{2003} + (\ln 4002)4002^x$

3. (16 points) In this problem you will work out the formula for $\frac{d}{dx}(\arcsin x)$. For parts (a) and (b), circle your answer.

(a) $\sin(\arcsin x) =$

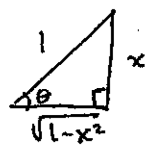
i. x ii. $\sqrt{1-x^2}$

iii. $\frac{1}{2}$ iv. $\frac{1}{\sqrt{1-x^2}}$

(b) $\cos(\arcsin x) =$

i. x ii. $\sqrt{1-x^2}$

iii. $\frac{1}{2}$ iv. $\frac{1}{\sqrt{1-x^2}}$



$\theta = \arcsin x$

$\sin\theta = x$

(Hint: You might find it useful to draw a triangle.)

(c) In part (a) you found the formula for $\sin(\arcsin x)$. Take the derivative of both sides of that formula with respect to x. Use your answer in part (b) to solve for $\frac{d}{dx}(\arcsin x)$. **For this part, assume no previous knowledge of** $\frac{d}{dx}(\arcsin x)$.

$$\sin(\arcsin x) = x$$

$$\cos(\arcsin x)\frac{d}{dx}[\arcsin x] = 1$$

$$\sqrt{1-x^2}\,\frac{d}{dx}[\arcsin x] = 1$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

4. (22 points) Let $f(x) = e^x$, $g(x) = x^2 + x$, and $h(x) = \frac{1}{x}$.

(a) What is $g(f(x))$?

$$g(f(x)) = (e^x)^2 + e^x = e^{2x} + e^x$$

(b) Write $k(x) = \frac{1}{e^{2x}+e^x}$ as a composition of f, g and (or) h.

$$\frac{1}{e^{2x}+e^x} = h(g(f(x)))$$

(c) What is $f(f(g(x)))$?

$$f(f(g(x))) = f(e^{x^2+x}) = e^{e^{x^2+x}}$$

(d) What is $\frac{d}{dx}(f(g(x)))$?

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[e^{x^2+x}] = (2x+1)e^{x^2+x}$$

5. (8 points) In this problem you will find a formula for $\frac{d}{dx}(x^x)$.

Let $y = x^x$.

Apply the natural logarithm to both sides to obtain the formula

$$\ln y = x \ln x.$$

(a) Take the derivative with respect to x of both sides of $\ln y = x \ln x$.

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

(b) Now write $\frac{dy}{dx}$ entirely in terms of x. Recall that we have a formula that relates y and x.

$$\frac{dy}{dx} = y(\ln x + 1)$$
$$= x^x(\ln x + 1)$$

This portion of the exam is True/False and Multiple Choice. Circle your answer. No partial credit will be given for incorrect answers.

6. (56 points) For this problem let

$$f(x) = \frac{1}{x^2 - 9}.$$

(a) What is the domain of f?

i. all real numbers

ii. all real numbers except 0

iii. all real numbers except 3

iv. all real numbers except 3 and -3

v. none of these

(b) Where is (are) the x-intercept(s) of f?

i. only at $(0, 0)$

ii. only at $(0, -\frac{1}{9})$

iii. only at $(-\frac{1}{9}, 0)$

iv. only at $(0, 0)$ and at $(-\frac{1}{9}, 0)$

v. none of these

(c) Where is (are) the y-intercept(s) of f?

i. only at $(0, 0)$

ii. only at $(0, -\frac{1}{9})$

iii. only at $(-\frac{1}{9}, 0)$

iv. only at $(0, 0)$ and at $(-\frac{1}{9}, 0)$

v. none of these

(d) Does f have any symmetry?

i. It is periodic.

ii. It is even.

$$f(-x) = \frac{1}{(-x)^2 - 9} = \frac{1}{x^2 - 9} = f(x)$$

iii. It is odd.

iv. It is even and periodic.

v. none of these

(e) Does f have any vertical asymptotes?

i. yes, only at $x = 3$ and $x = 0$

ii. yes, only at $x = 3$

iii. yes, only at $x = 3$ and $x = -3$

iv. yes, only at $x = 0$

v. none of these

(f) Does f have any horizontal asymptotes?

i. yes, only at $y = 3$ and $y = 0$

ii. yes, only at $y = 3$

iii. yes, only at $y = 3$ and $y = -3$

iv. yes, only at $y = 0$

v. none of these

$$\lim_{x \to \infty} \frac{1}{x^2 - 9} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x^2 - 9} = 0$$

$f(x) = \frac{1}{x^2-9}$

$f'(x) = \frac{-2x}{(x^2-9)^2}$

+ + − −

−3 0 3

(g) On what interval(s) is f increasing?

i. only on $(-\infty, -3)$ and on $(0, \infty)$

ii. only on $(-3, 0)$ and on $(0, 3)$

iii. only on $(-\infty, -3)$ and on $(-3, 0)$

iv. only on $(-\infty, -3)$ and on $(0, 3)$

v. none of these

(h) Does f have any local maxima or local minima?

i. Yes, it has a local maximum at $x = 0$.

ii. Yes, it has a local minimum at $x = 0$.

iii. Yes, it has a local maximum at $x = 0$ and local minima at $x = -\frac{1}{9}$ and at $x = -3$.

iv. It has no local maxima or local minima.

v. none of these

(i) On what interval(s) is f concave down?

i. only on $(-\infty, -3)$ and on $(-3, -\frac{1}{9})$

ii. only on $(-3, 0)$ and on $(0, 3)$

iii. only on $(-\infty, -3)$ and on $(3, \infty)$

iv. only on $(-3, 3)$

v. none of these

$$f''(x) = \frac{-2(x^2-9)^2 - 2(x^2-9)2x(-2x)}{(x^2-9)^4}$$

$$= \frac{-2(x^2-9) + 8x^2}{(x^2-9)^3}$$

$$= \frac{6x^2+18}{(x^2-9)^3}$$

$$= \frac{6(\cancel{3}x^2+3)}{(x^2-9)^3}$$

+ − +

−3 $\cancel{0}$ 3

(j) Does f have any inflection points?

i. yes, only at $x = 3$

ii. yes, only at $x = -\frac{1}{9}$ and $x = -\frac{1}{9}$

iii. yes, only at $x = 0$

iv. no inflection points

or v. none of these

7. (6 points) Suppose the function $y = g(x)$ has the following properties:

(a) the domain of g is all real numbers except $x = -2$ and $x = 2$;

(b) g is an even function;

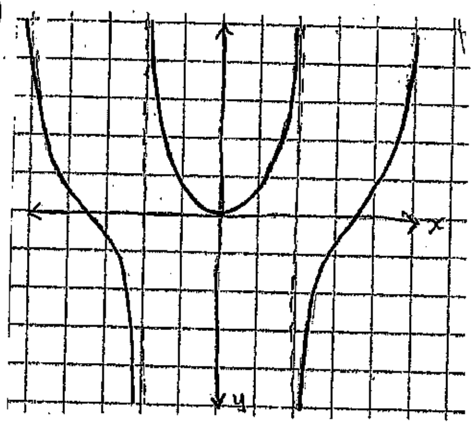
(c) g is *decreasing* on $(-\infty, -2)$ and on $(-2, 0)$;

(d) g has no local maxima, but g *does* have a local minimum at $x = 0$;

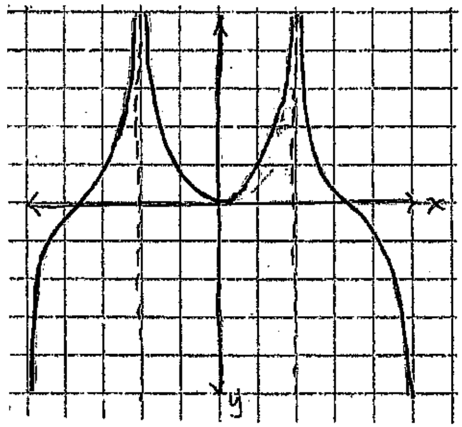
(e) g is *increasing* on $(0, 2)$ and on $(2, \infty)$; and

The graph of $y = g(x)$ looks like (circle your answer):

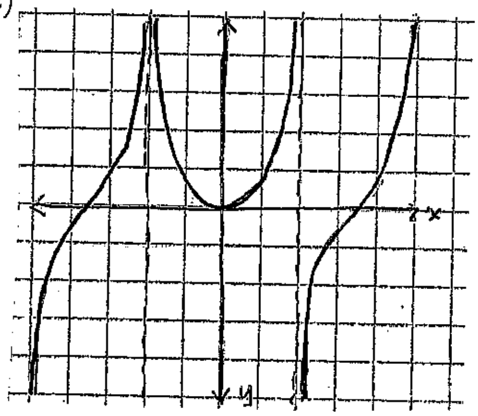
(a)



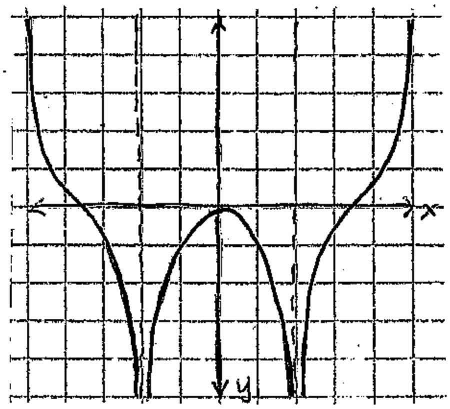
(b)



(c)



(d)



(e) none of these

8. (12 points) If $f' = g'$ everywhere on their domains, what must always be true?

(a) $f'' = g''$ everywhere on their domains. True False

(b) $f = g$ everywhere on their domains. True False

9. (18 points) If $f'' > 0$ on the interval (a, b) then it is always true that

(a) f must be increasing on (a, b). True False

(b) f must be concave up on (a, b). True False

(c) f must be positive on (a, b). True False

10. (27 points) Are the following identities true?

(a) $\cos^2 x - \sin^2 x = 1$ True False

(b) $\cos^{-1} x = \sec x$ for every real number x True False

(c) $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ True False

(d) $\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x)$ True False

(e) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x$. True False

Scratch work here.