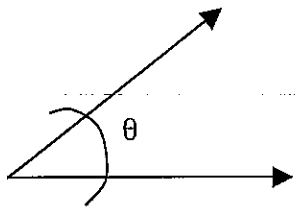
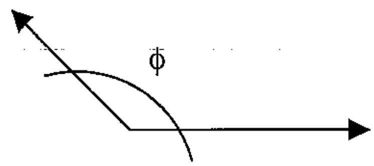
Trigonometry 1

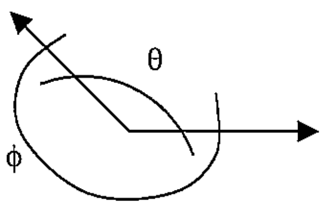
One of the most practical areas of Pre-Calculus mathematics is Trigonometry. People discovered the use of trigonometry in astronomy many years before anyone had discovered Calculus. For many years scientists didn't even distinguish between the two subjects. Now days, Trigonometry is used for everything from satellite technology, engineering, computer graphics, and more.

An **angle** is the figure formed by two rays sharing a common endpoint, called the vertex of the angle. The greek letter θ, pronounced theta, is traditionally used to represent the angle. Other strange letters you may see are ϕ (phi), ψ (psi), and ρ (rho).





Traditionally we look at angles on the coordinate plane. We fix a ray on the x-axis and let the other ray sweep out. If the ray sweeps out counterclockwise, we say the angle is **positive**. If the ray sweeps out clockwise, the angle is **negative**. Be careful with this because it is sometimes counterintuitive!!!



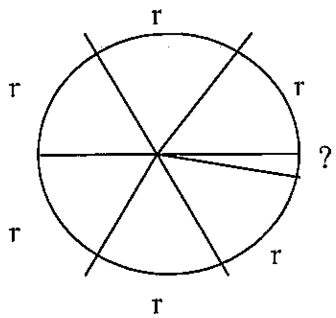
Here, θ is the positive angle and ϕ is the negative angle.

There are two ways to measure an angle: Degrees and Radians.
We will mostly use radians in this course because it is the standard in science.
For completeness we will review both.

A circle contains 360 degrees. One **degree** thus represents 1/360th of a complete rotation. A **right angle** is 90 degrees.
The problem with degrees is the arbitrariness of the definition of degree. We don't have a good notion of why a degree is what I just told you it is. This is why scientists turned to radians.

Radians relate the circumference of a circle with the angle. One **radian** represents the angle whose arc on a circle is equal in length to the radius of the circle.

The circumference of a circle with radius r we remember is $2\pi r$. If we were to walk around a circle of radius r and drop a pebble every r units we would end up with the following:



Notice that there are 6 pebbles dropped r units apart but one more pebble which is dropped at the end of the trip around.

What is the distance from the last pebble we dropped to the first pebble? It turns out that 2π is approximately 6.28. So in one trip around a circle of radius r, we travel 6r units and then approximately 0.28r units. The mystery distance at the end is about 0.28r units. One radian is then $1/2\pi$.

To convert between radians and degrees, note that 360 degrees = 2π radians. Now it becomes a ratio problem.

Example: Convert 60 degrees to radians.

$$\frac{360 \text{ degrees}}{60 \text{ degrees}} = \frac{2\pi \text{ radians}}{x \text{ radians}}$$

So
$$360x = 60 * 2\pi$$
$$360x = 120\pi$$
$$x = 120\pi / 360$$
$$x = \pi/3 \text{ radians.}$$

Example: Convert $\pi/4$ radians to degrees.

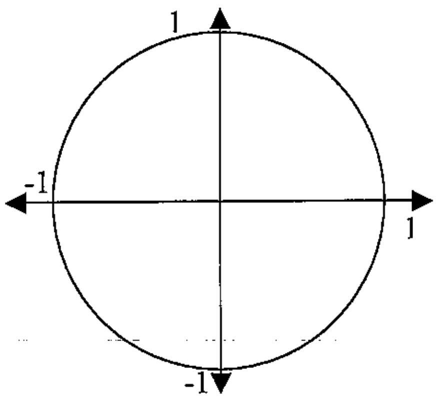
$$\frac{360 \text{ degrees}}{x \text{ degrees}} = \frac{2\pi \text{ radians}}{\pi/4 \text{ radians}}$$

So
$$360 * \pi/4 = 2\pi x$$
$$90\pi = 2\pi x$$
$$90\pi / 2\pi = x$$
$$x = 45 \text{ degrees.}$$

Here are the most important angles for this class.

Degrees	0	30	45	60	90	180	360
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π

To learn about triangles we will begin with circles.
Usually we will play with circles of radius 1, centered at the origin.

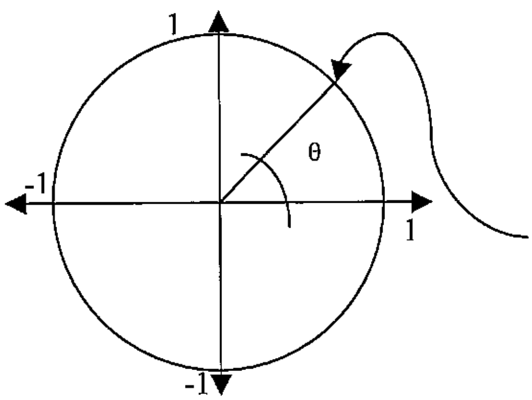


We will call it the **unit circle**.

It will become your best friend in trigonometry. With it you will not need to memorize calculations and you will not need to depend on your calculator anymore.

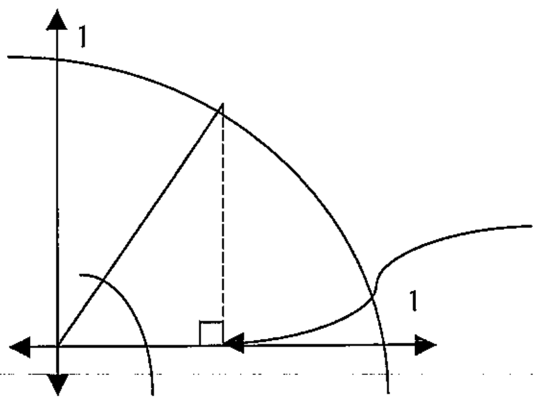
We will always put one ray of our angle on the positive x-axis with the vertex on the origin.
The other ray is allowed to swing around.

The question we would like to answer is *where is the second ray crossing the circle*?



What are the exact coordinates for this point?

To find the exact location of the point given a certain θ let's zoom in a bit closer.



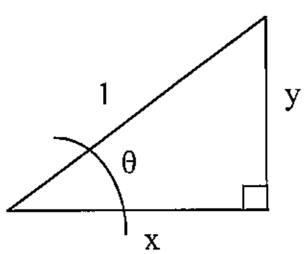
Let's first find the x-coordinate. We want the value directly below the point. I drew a dotted line to show where the point is.

It is perpendicular (forms a right angle) to the x-axis.

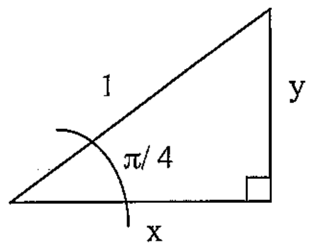
Now we have a triangle. It turns out to be a right triangle (one angle is $\pi/2$). The hypotenuse of our triangle is the same as the radius of the circle, namely 1. Right triangles are nice because we have the **Pythagorean Theorem**. It says that a right triangle with sides of length x and y and hypotenuse of length z satisfies $\mathbf{x^2 + y^2 = z^2}$.

The Pythagorean Theorem will be very useful for the rest of this chapter.

For now, our triangles all look like this:



Let's try to figure out (x , y) for some specific values of θ.
Let $\theta = \pi/4$.



The angles of a triangle must add up to π and we know one angle has π/ 4 radians and one has π/ 2. And we know that $\pi - \pi/4 - \pi/2 = \pi/4$. Therefore, the other angle must be π/ 4 as well. So we have an isosceles triangle with the sides x and y the same.

Now we use the Pythagorean Theorem.

$x^2 + y^2 = 1^2$ but since x = y

$x^2 + x^2 = 1$

$2x^2 = 1$

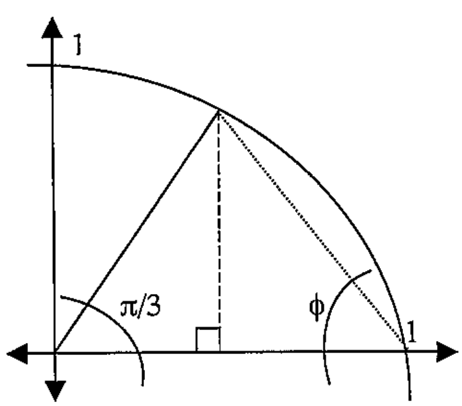
$x^2 = 1/2$

$x = 1/\sqrt{(2)} = \sqrt{(2)}/2.$

So x and y are both equal to $\sqrt{(2)}/2$.

That went pretty well. Let's try another one.

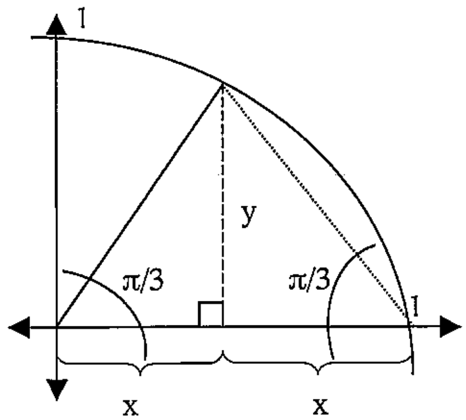
Let $\theta = \pi/3$.



Draw a line from (x , y) to (1 , 0). We have a new big triangle. Two sides have length 1; the two sides with endpoints at the origin.

We don't yet know the length of the other side.

Since two sides of the triangle are the same, their opposite angles must be the same. We use the fact that all the angles of a triangle must add up to π. So we have $\pi - \phi - \phi = \pi/3$ which gives us $\phi = \pi/3$.



This means our triangle is equilateral (all the sides have the same length). In an equilateral triangle, if you drop a perpendicular from any of the vertices (as we did with our dotted line) it will bisect the base of the triangle.
Here we have that the base is 2x.

But we already knew that the base of our triangle is 1. So $2x = 1$ or, $x = 1/2$.
Now that we know what x is, we can use the Pythagorean Theorem to find what y is.

$x^2 + y^2 = 1^2$ but since $x = 1/2$

$(1/2)^2 + y^2 = 1$

$1/4 + y^2 = 1$

$y^2 = 3/4$

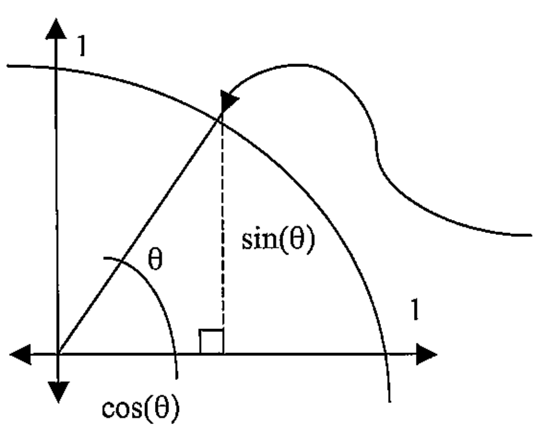
$y = \sqrt{(3/4)} = \sqrt{(3)}/2.$

So we have found (x, y) is $(1/2, \sqrt{(3)}/2)$ when θ is $\pi/3$.

It will turn out that $\theta = \pi/6$ will give us similar results. Can you see why?

When θ is $\pi/2$, clearly $x = 0$ and $y = 1$.
When θ is π, clearly $x = -1$ and $y = 0$.

If you haven't recognized what just happened, we have just constructed $\sin(\theta)$ and $\cos(\theta)$ as people used to do it before the invention of calculators. Notice how nowhere in here did I say, "Now plug this into your graphing calculator…" or "Now use your calculator to compute…"

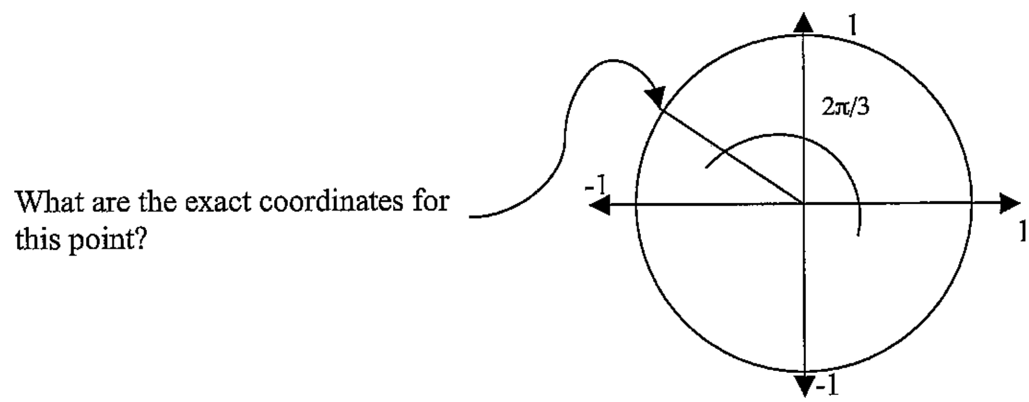


It turns out that the point (x , y) we have been calculating is actually $(\cos(\theta), \sin(\theta))$.

So far we have derived the following computations:

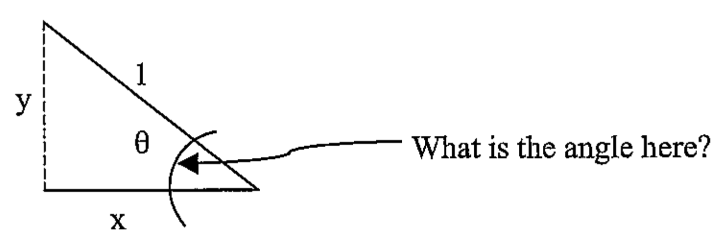
Degree	0	$\pi/4$	$\pi/3$	$\pi/6$	$\pi/2$	π
$\sin(\theta)$	0	$\sqrt{(2)}/2$	1/2	$\sqrt{(3)}/2$	1	0
$\cos(\theta)$	1	$\sqrt{(2)}/2$	$\sqrt{(3)}/2$	1/2	0	-1

What about $2\pi/3$?



Just like before, we drop a perpendicular down from our point to the x-axis.

This is the triangle from above recopied in larger scale.



If you calculated that the angle was $\pi/3$, you are right! Now we have reduced the problem to one we already know how to solve. We know that for $\theta = \pi/3$, x will be $\sqrt{(3)}/2$ and y will be 1/2. But notice here that the point we are looking for is not at

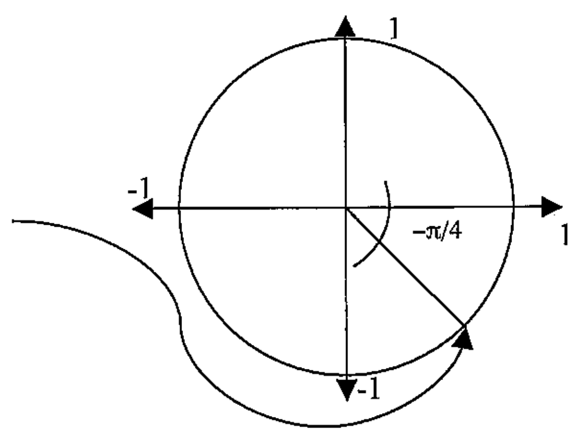
(√(3)/2 , 1/2). Its x coordinate is negative! This point is at (-√(3)/2 , 1/2).

Once again, we didn't need a calculator. We didn't need to memorize tables with radians and degrees and values of sine and cosine. We can derive everything!

Let's try one more.

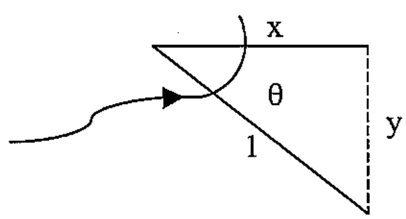
What about $-\pi/4$?

What are the exact coordinates for this point?



This time, when we "drop" a perpendicular from our point to the x-axis, it will actually be going up.

What is the angle here?



This is the triangle from above recopied in larger scale.

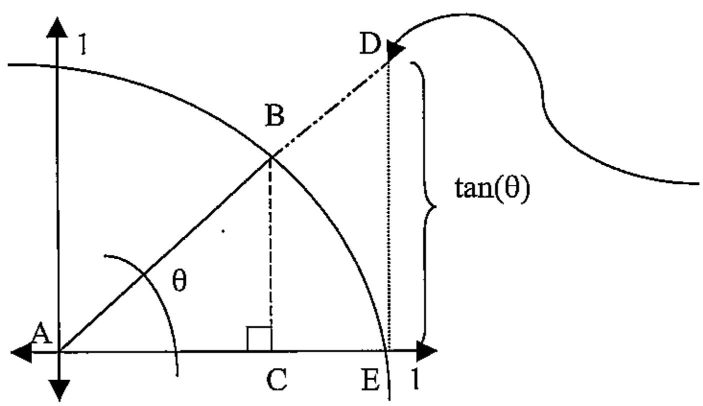
If you calculated that the angle was $\pi/4$, you are right! Again, we have reduced the problem to one we already know how to solve. We know that for $\theta = \pi/4$, x will be

√(2)/2 and y will be √(2)/2. But notice here that the point we are looking for is not at (√(2)/2 , √(2)/2). Its y coordinate is negative! This point we want is at (√(2)/2, -√(2)/2).

No tables, calculators, super computers necessary! All you need is a unit circle and a little elbow grease.

You may recall the existence of another major trig function, the tangent function. How does tan(θ) fit into the unit circle?

First let's review what the word tangent means. A line is **tangent** to a circle if it touches the circle at exactly one point. It would make sense then for the tangent function to involve a line which is tangent to the unit circle.



Draw a line perpendicular to the x-axis and tangent to the circle at (1, 0). Now extend the ray until it meets our perpendicular line. The y value of this new point is tan(θ).

How do we calculate such a value? Did you notice the similar triangles?

Since the two triangles ADE and ABC are similar, we have

$$ED / AE = CB / AC.$$

But ED = tan(θ), AE = 1, CB = sin(θ), and AC = cos(θ). Therefore we have derived the fundamental identity.

$$\tan(\theta) = \sin(\theta) / \cos(\theta).$$

So calculating the tangent for the familiar angles should be routine.

Degree	0	π/6	π/4	π/3	π/2	π
sin(θ)	0	√(3)/2	√(2)/2	1/2	1	0
cos(θ)	1	1/2	√(2)/2	√(3)/2	0	-1
tan(θ)	0	√(3)	1	√(3)/3	∞	0

You may have noticed that for $\pi/2$, $\tan(\theta)$ is infinity. This is because $\cos(\pi/2) = 0$ and as you know dividing by 0 is not allowed. When we look at the graphs of our three new functions you will see how the function behaves near $\pi/2$.

When we are trying to learn about a function we have discussed three ways of examining it; Verbally, Numerically and Graphically.
So far we have examined our trigonometric functions verbally. We know their geometric interpretation using the unit circle.
We have examined them numerically for a few select values. In the process we made some tables showing the values we calculated.
We know that verbal and numerical descriptions can only tell us a limited amount about most functions. What we really want to do now is to look at the graphs.

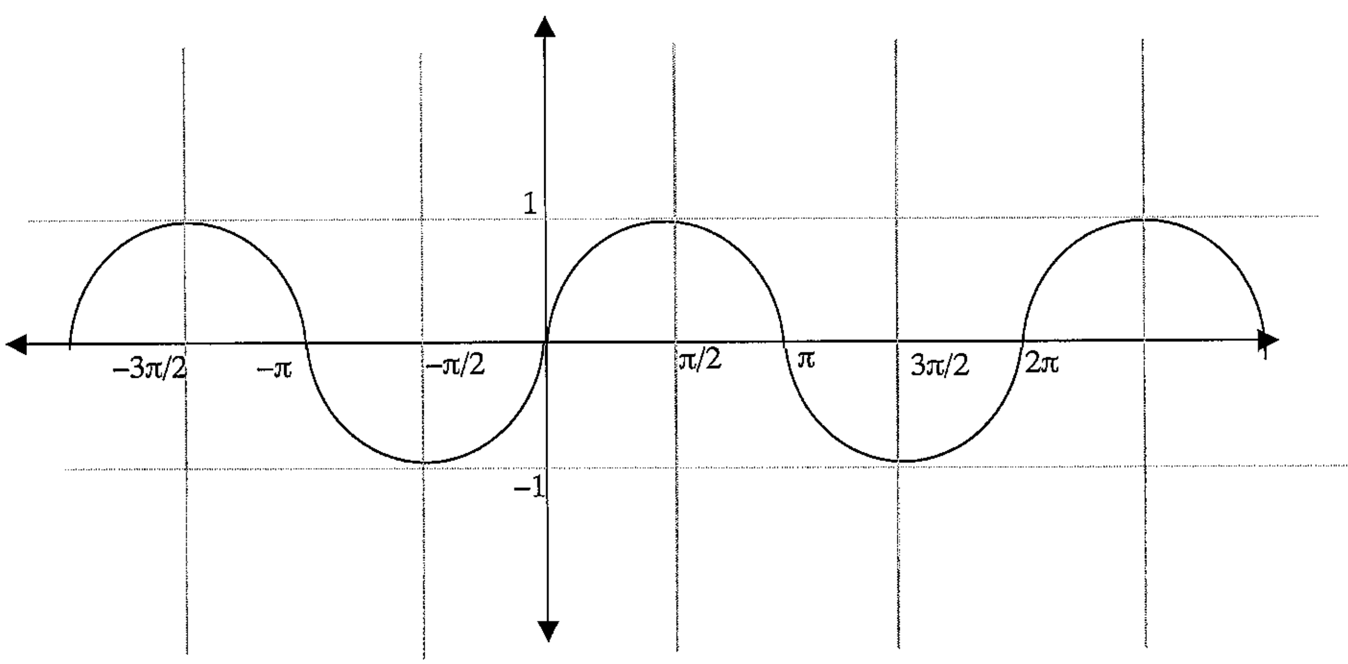
A function f(x) is **periodic** if there is a real number p such that f(x) = f(x + p) for every x in the domain. We say that the **period** of f(x) is p.

Periodic functions are easy to graph because they repeat. If you know the value at f(x) then you also know the value at f(x + p) and f(x + 2p) and f(x + 3p) and so on....

The **frequency** of a periodic function is the reciprocal of the period (i.e. 1/period).

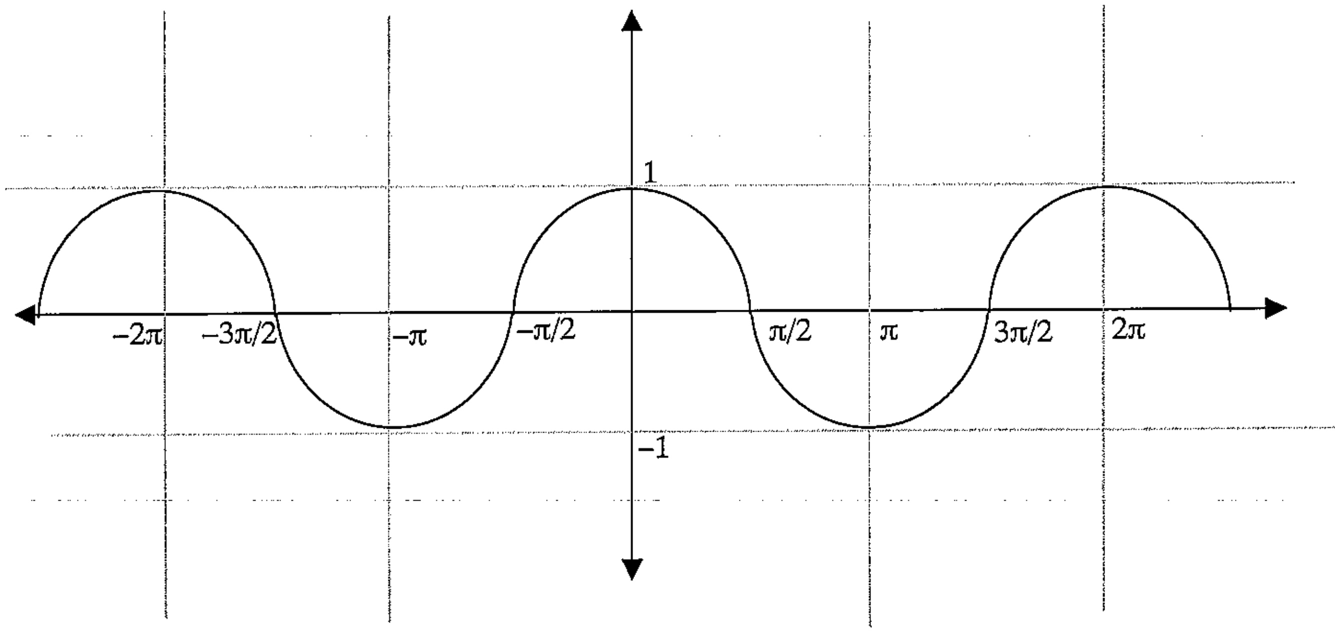
The **amplitude** of a periodic function is half the distance from the highest y-value to the lowest y-value.

Let's start with the graph of $\sin(\theta)$.



Notice how the graph repeats. What do you think the period of this graph is? In other words, when does the graph begin repeating itself? Hint: it is not π.

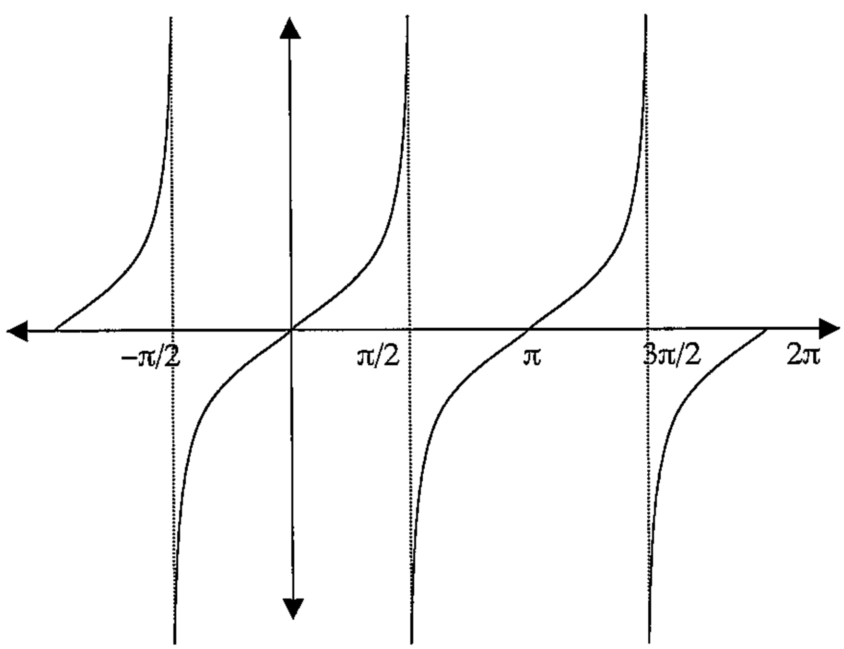
Let's move on to the graph of cos(θ).



What is the period of this graph? What is the amplitude?

Finally, let's look at the graph of tan(θ).

Notice how at π/2 the function marches off to infinity and negative infinity depending on which direction you approach.



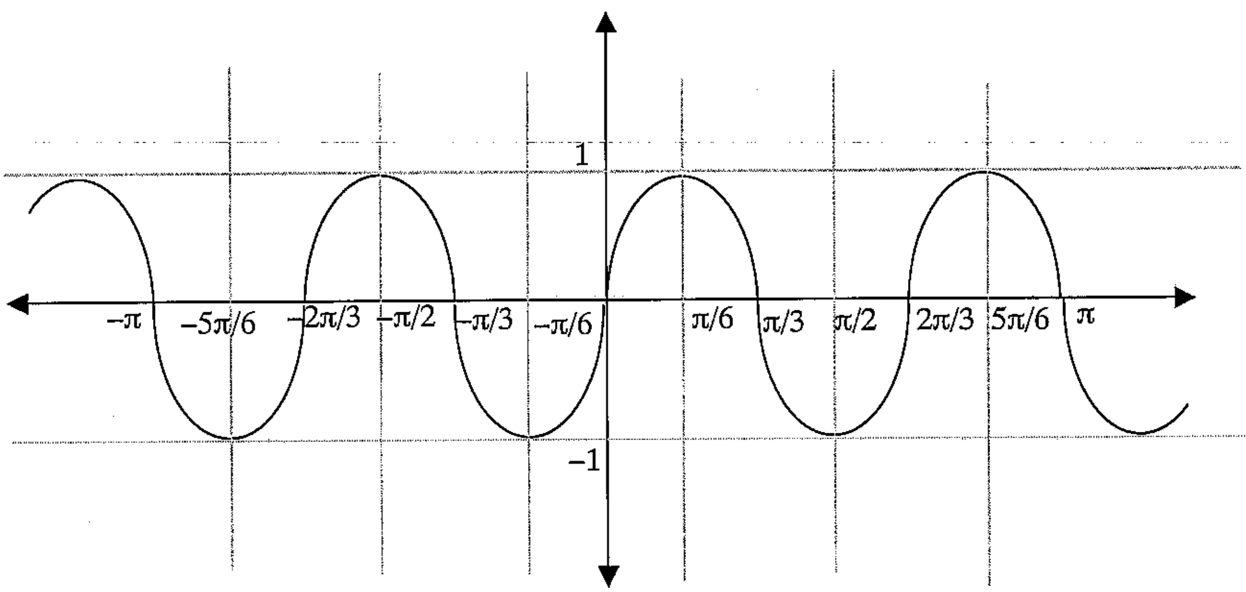
What is the period of this graph? Hint: it is not 2π.

What might the graph of sin(3θ) look like?

Let's start by plugging in some values.

θ	0	π/3	π/6	π
3θ	0	π	π/2	3π
sin(3θ)	0	0	1	0

It seems that the graph should look something like this:



Notice how it looks just like the sine graph only it has a higher frequency. What is its period? What is its amplitude?

It turns out that multiplying the angle of a sine, cosine or tangent function by some number n will multiply the period by 1/n.

What is the period of cos(5 θ)?
What is the frequency of sin(100 θ)?

To change the amplitude of a sine or cosine function, just multiply the function by the new amplitude.

For example, the amplitude of 10sin(θ) is 10.

This will not change the period of the function.