Consider the function $\sin\theta$**. Find a function** $f(x)$ **so that** $f(\sin\theta) = \theta$ **and** $\sin f(x) = x$**. (A verbal description is sufficient; you need not supply an explicit formula.) Consider domain, range, and codomain. Why did you pick this function? Why might having such a function be useful?**

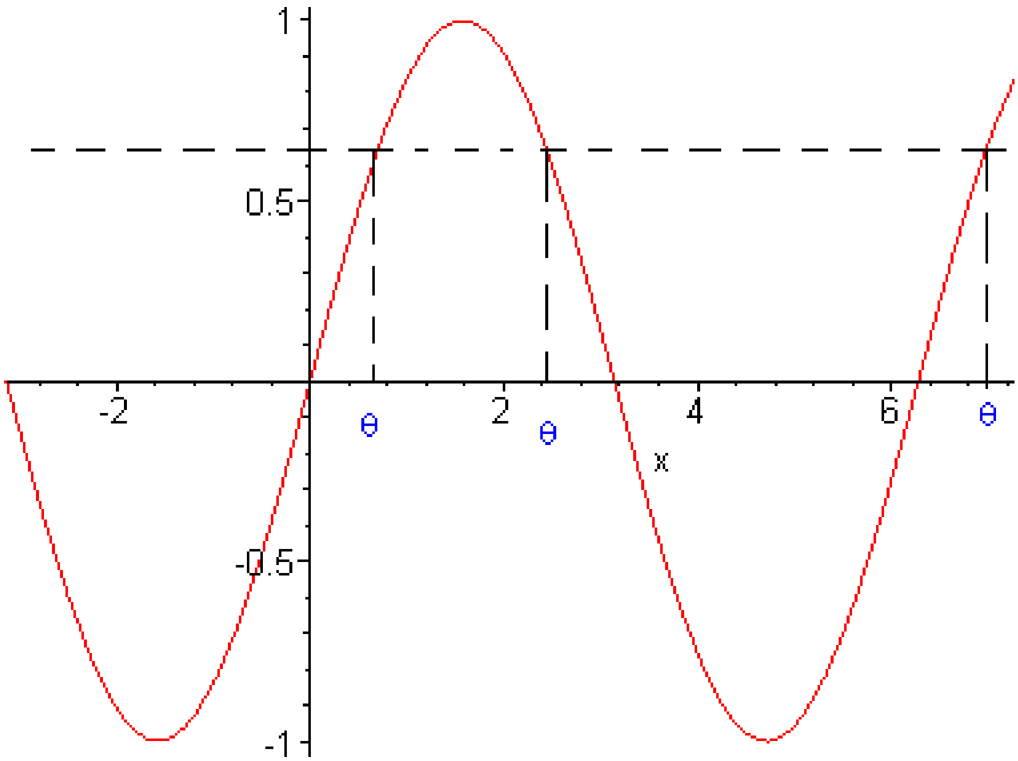

Figure 1: A graph of $\sin\theta$ demonstrating that this function does not map each input to a unique output.

Consider a right triangle where you are given the lengths of the sides. It would be nice to find the angle between one of the legs and the hypotenuse. Going the other way, from the angle to the length of the side, we can use $\sin\theta$, where θ is the angle. The problem is to find a functional inverse of $\sin\theta$.

In its most general form, $\sin\theta$ maps all real numbers to a value between -1 and 1. The best domain we could hope for in the inverse function is the range of the the original function, in this case $[-1, 1]$. Unfortunately, this domain of $\sin\theta$ cannot be the same as the range of the inverse since there are many different values of θ which give the same value for $\sin\theta$, as seen in Figure 1. Even if we take only the domain $[0, 2\pi)$, the inputs do not give unique outputs (see Figure 2). On this second domain, the first repetition of output values occurs after $\pi/2$, but if we consider the interval $[0, \pi/2]$ as our domain, there are some values in the original range that aren't hit (see Figure 3). Since this is the range that is desired for the domain of the inverse function, it is hoped that we could do better. It seems natural to include 0 in the

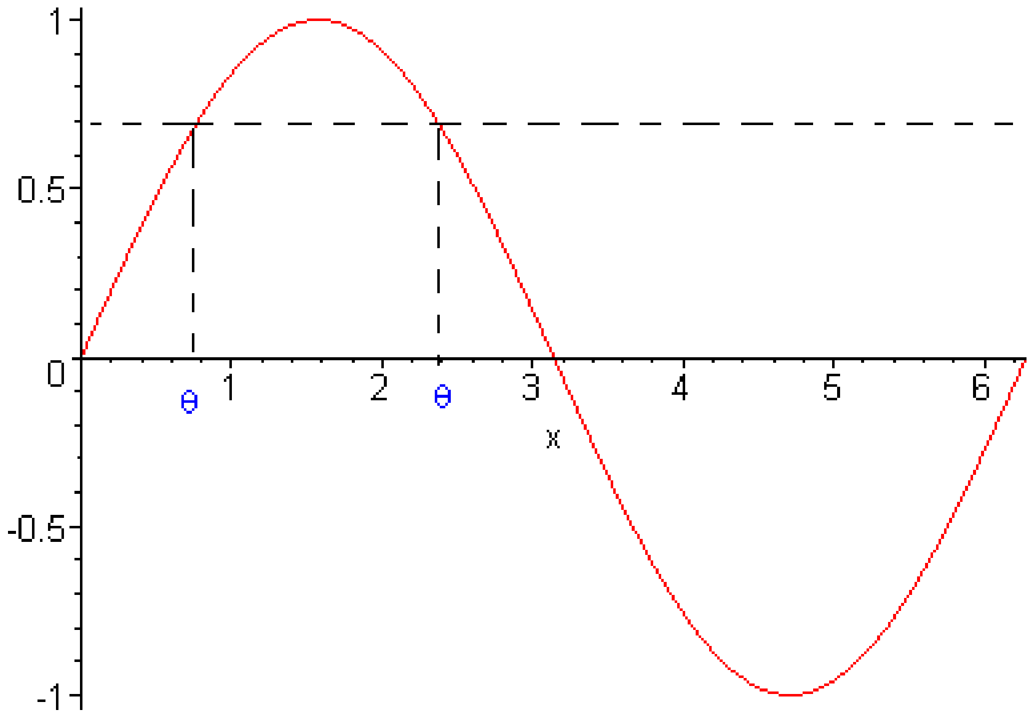

Figure 2: $\sin\theta$ with domain $[0, 2\pi)$. It still does not give a unique output.

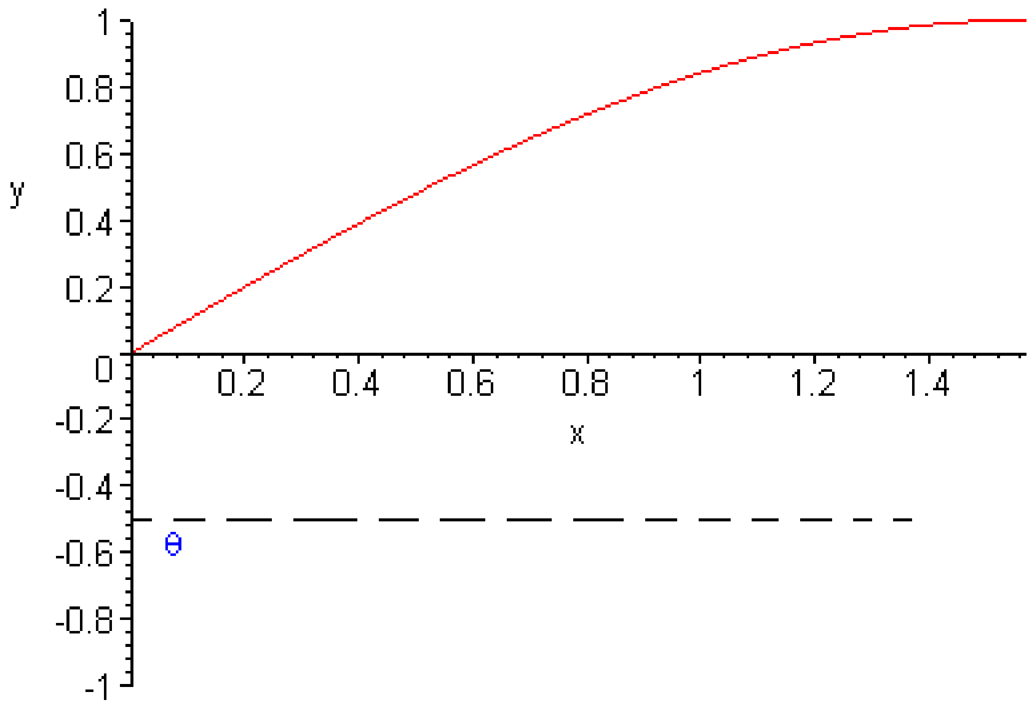

Figure 3: $\sin\theta$ on $[0, \pi/2]$. Notice that there is no input which gives output θ.

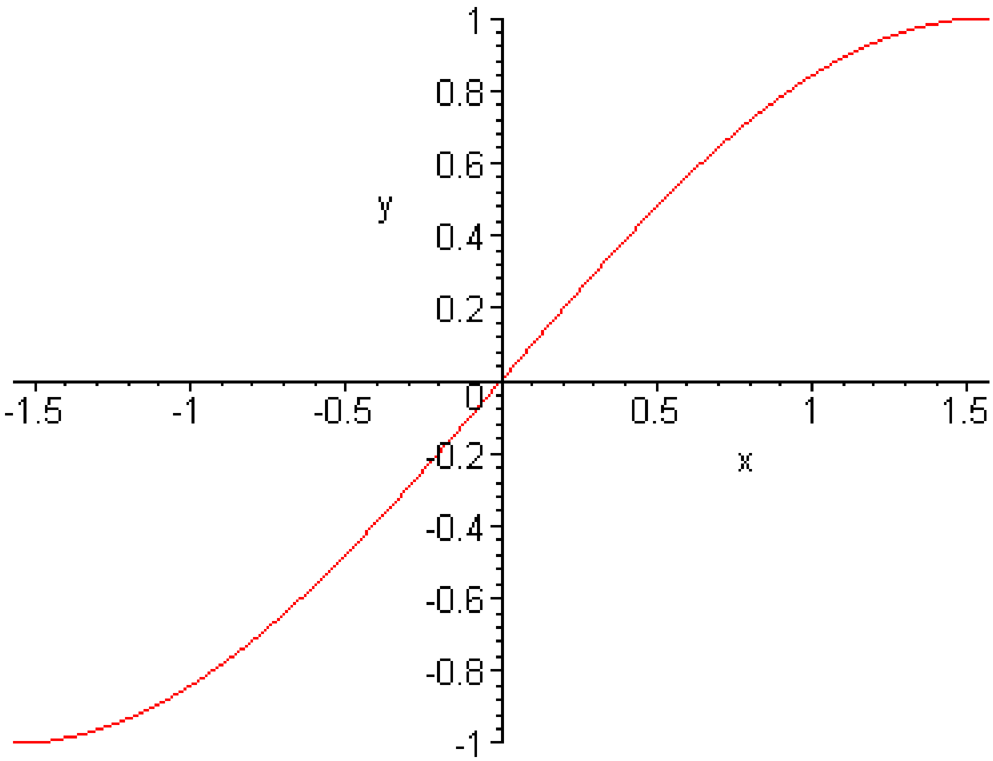

Figure 4: $\sin\theta$ on $[-\pi/2, \pi/2]$. There is no repetition of outputs.

domain of $\sin\theta$, that the largest interval around 0 without repetition of outputs is $[-\pi/2, \pi/2]$ (see Figure 4). This also happens to cover the entire range $[-1, 1]$, so this would be a good choice for the function to invert. The easy way to invert the function is to graph θ on the y-axis and $\sin\theta$ on the x-axis (so the points $(\sin\theta, \theta)$ are on the graph) and, for any input x, output y so that (x, y) is on the graph. The graph of this function is Figure 5. The domain of this function is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.

It remains only to check that the new function, which we call f, has the properties of an inverse function. Specifically, $f(\sin\theta) = \theta$ and $\sin f(x) = x$. Suppose $f(x) = \theta$ so (x, θ) is on the graph of f (Figure 5). Then (θ, x) is on the graph of sin (Figure 4) and $\sin\theta = x$. This gives $\sin f(x) = x$. Showing $f(\sin\theta) = \theta$ is similar, starting with the supposition that $\sin\theta = x$, giving (θ, x) on the graph of sin an (x, θ) on the graph of f.

This gives f on $[-1, 1]$ as the inverse to sin on $[-\pi/2, \pi/2]$.

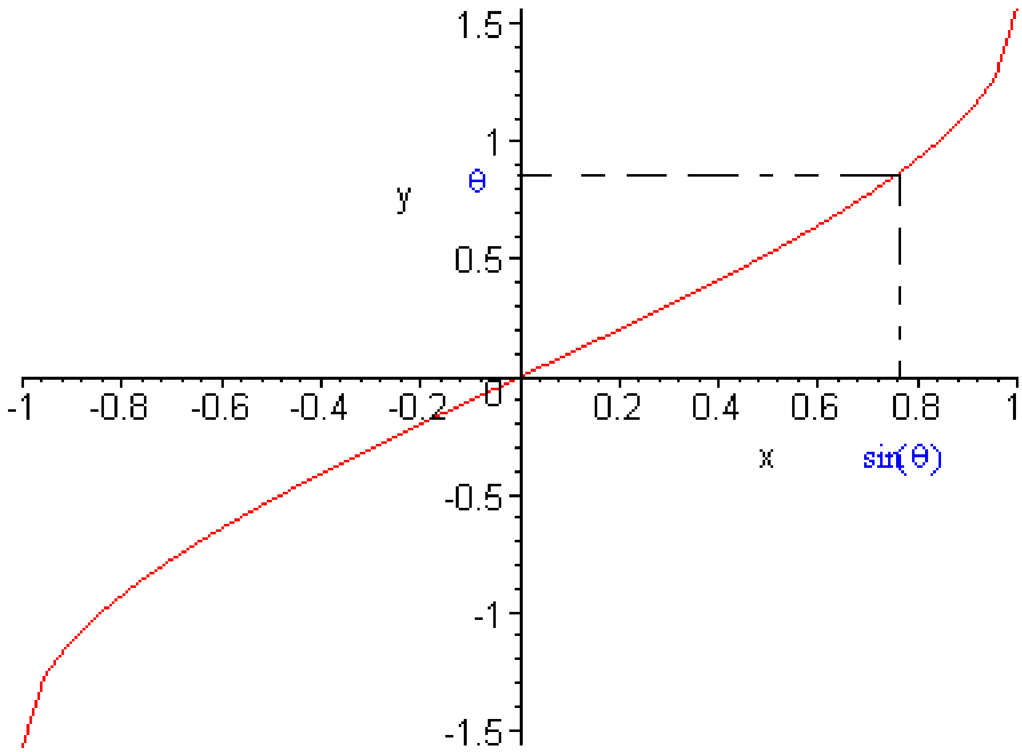

Figure 5: A graph of the inverse of $\sin\theta$. Call this function f.