MATH 1 Homework 5 Solutions

Assigned October 12th, due October 19th

1. (a) To have $\frac{1}{(x-1)^4} > 10000$, we need

$$\frac{1}{10000} > (x-1)^4$$
$$\sqrt[4]{\frac{1}{10000}} > |x-1|$$
$$\frac{1}{10} > |x-1|,$$

so x needs to be at most 0.1 away from 1.

(b) In this case,

$$\frac{1}{160000} > (x-1)^4$$
$$\sqrt[4]{\frac{1}{160000}} > |x-1|$$
$$\frac{1}{20} > |x-1|,$$

so x needs to be at most $1/20 = 0.05$ away from 1.

(c) Since the value of f gets larger and larger as x approaches 1, $\lim_{x\to 1} f(x) = +\infty$.

2. Let

$$f(x) = \begin{cases} x & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$$

and $g(x) = x+2$. Then $\lim_{x\to 1} g(x) = \lim_{x\to 1} x+2 = 3$. But $\lim_{x\to 1} f(g(x)) = \lim_{x\to 1} f(x+2) = \lim_{x\to 1} x+2 = 1+2 = 3$, while $f(3) = 0$.

3. Let $f(x) = \frac{x+1}{x-3}$. Then $f(5) = 6/2 = 3$, f has a vertical asymptote at $x = 3$, and a horizontal asymptote at $y = 1$.

4. (a) We can take any delta $\delta < 0.07$ (rough approximation; we might be able to take δ a bit lager than that, but for $\delta < 0.07$, it is guaranteed to work)

(b) If we take $\delta < 0.08$, then $|g(x) - 1.5| < 0.2$. If we take any delta $\delta < 0.04$, then $|g(x) - 1.5| < 0.1$ (again, a rough approximation based on the graph).

5. The floor function, ceiling function, and $\tan(x)$ all have infinitely many discontinuities: the floor and ceiling functions both have jump discontinuities at every integer, and $\tan(x)$ has asymptotes at $k\pi/2$ where k is an integer.

6. (a) $f(x) = \cos\left(\frac{x+3}{x^2-2}\right)$ at $x = 5$. Yes - rational functions are continuous everywhere in their domain, and 5 is in the domain of $\frac{x+3}{x^2-2}$. $\cos(x)$ is continuous everywhere. Thus their composition is continuous.

(b) $g(x) = \tan\left(\frac{x-\frac{\pi}{2}}{x-\pi+1}\right)$ at $x = \pi$. Plugging in $x = \pi$ to the inner function yields $\frac{\pi}{2}$. But $\pi/2$ is not in the domain of tan. Therefore the function is not continuous.

(c) $h(x) = \ln(x^2 - 3)$ at $x = 2$. $2^2 - 3 = 1$ is in the domain of ln, which is a continuous function on its domain. Thus this is continuous.

(d) $k(x) = 2^{\log_3(\sqrt{x})}$ at $x = 17$. $\sqrt{17}$ is in the domain of $\log_3$, so $\log_3(\sqrt{x})$ is continuous at 17. 2^x is continuous everywhere. Thus their composition is continuous.

7. (a) $\lim_{x\to 0} x^2 \tan^{-1}(\frac{1}{x})$. $-\pi/2 \le \tan^{-1}(\frac{1}{x}) \le \pi/2$, so $-\frac{\pi x^2}{2} \le \tan^{-1}(\frac{1}{x}) \le \frac{\pi x^2}{2}$. If we plug in zero, we get 0 on both sides of the inequality. Thus the limit is zero.

(b) $\lim_{x\to 0} xe^{\sin(\frac{1}{x})}$. We know that $-1 \le \sin(\frac{1}{x}) \le 1$, so raising e to all sides yields $e^{-1} \le e^{\sin(\frac{1}{x})} \le e$.

To be able to use the squeeze theorem, we want to multiply throgh by x. However, depending on whether $x \ge 0$ or $x < 0$, we have to consider two cases: multiplying through by a positive number keeps the inequality signs unchanged, while multipliyng by a negative number reverses the signs.

Case 1: $x \ge 0$. Then multiplying through by x gives $xe^{-1} \le xe^{\sin(\frac{1}{x})} \le xe$. Then $\lim_{x\to 0^+} xe^{-1} = 0$, $\lim_{x\to 0^+} xe = 0$, so $\lim_{x\to 0^+} xe^{\sin(\frac{1}{x})} = 0$.

Case 2: $x < 0$. Then multiplying through by x gives $xe^{-1} \ge xe^{\sin(\frac{1}{x})} \ge xe$. Then $\lim_{x\to 0^-} xe^{-1} = 0$, $\lim_{x\to 0^-} xe = 0$, so $\lim_{x\to 0^-} xe^{\sin(\frac{1}{x})} = 0$.

Since both one- sided limits $\lim_{x\to 0^+} xe^{\sin(\frac{1}{x})} = 0$, $\lim_{x\to 0^-} xe^{\sin(\frac{1}{x})} = 0$, then $\lim_{x\to 0} xe^{\sin(\frac{1}{x})} = 0$.

(c) $\lim_{x\to -3} (x+3)\cos(\frac{1}{x+3})$. We know that $-1 \le \cos(x) \le 1$, so $-1 \le \cos(\frac{1}{x+3}) \le 1$.
As above, we have two cases: when $x \ge -3$ and when $x < -3$.

Case 1: $x \ge -3$. Then multiplying through by $x-3$ gives $-(x+3) \le (x+3)\cos(\frac{1}{x+3}) \le (x+3)$. Then $\lim_{x\to -3^+} -(x+3) = 0$, $\lim_{x\to -3^+} (x+3) = 0$, so $\lim_{x\to -3^+} (x+3)\cos(\frac{1}{x+3}) = 0$.

Case 2: $x < -3$. Then multiplying through by $x-3$ gives $-(x+3) \ge (x+3)\cos(\frac{1}{x+3}) \ge (x+3$. Then $\lim_{x\to -3^-} -(x+3) = 0$, $\lim_{x\to -3^-} (x+3) = 0$, so $\lim_{x\to -3^-} (x+3)\cos(\frac{1}{x+3}) = 0$.

Since both one-sided limits are 0, $\lim_{x\to -3} (x+3)\cos(\frac{1}{x+3}) = 0$.

8. (a) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$. We can multiply top and bottom by $\sqrt{x+3}+\sqrt{3}$ to get $\frac{x+3-3}{x(\sqrt{x+3}+\sqrt{3})}$. We can cancel the x's to get $\frac{1}{\sqrt{x+3}+\sqrt{3}}$. To find the limit, we can now plug in 0 (since this is continuous and 0 is in the domain) to get that the limit is $\frac{1}{2\sqrt{3}}$.

(b) $\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2}$. We can multiply top and bottom by $\sqrt{4x+1}+3$ to get $\frac{4x+1-9}{(x-2)(\sqrt{4x+1}+3)}$. The numerator is $4x-8 = 4(x-2)$, so the fraction can be rewritten as $\frac{4(x-2)}{(x-2)(\sqrt{4x+1}+3)}$. If we cancel the $x-2$, we get $\frac{4}{\sqrt{4x+1}+3}$. This is a continuous function, and 2 is now in its domain, so we can plug it in to find that the limit is $\frac{4}{\sqrt{4(2)+1}+3} = \frac{4}{3+3} = \frac{2}{3}$.