Newton's Method Practice

1. Consider the function
$$x^5 - x^3 + 2x^2 - 1$$
Approximate the root near 1 by eight decimal places.

   Answer:

   Our function is $f(x) = x^5 - x^3 + 2x^2 - 1$. Our initial point will be $x_1 = 1$. The derivative of the function is $f'(x) = 5x^4 - 3x^2 + 4x$. Thus to compute the next approximation, we use the formula
$$x_{n+1} = x_n - \frac{x_n^5 - x_n^3 + 2x_n^2 - 1}{5x_n^4 - 3x_n^2 + 4x_n}$$
And so we get
$$\begin{aligned} x_1 &= 1 \\ x_2 &\approx 0.83333333 \\ x_3 &\approx 0.77541271 \\ x_4 &\approx 0.77005822 \\ x_5 &\approx 0.77001784 \\ x_6 &\approx 0.77001784 \end{aligned}$$
Since we have repeated the first eight decimal places, we have that $f(x)$ has a root at $x \approx 0.77001784$.

2. Find the 10^{th} root of 3 to four decimal places.

   Answer:

   We know that the 10^{th} root of 3 is a root to the function $f(x) = x^{10} - 3$.

   Thus, our function is $f(x) = x^{10} - 3$. Our initial point will be $x_1 = 1$. The derivative of the function is $f'(x) = 10x^9$. Thus to compute the next approximation, we use the formula
$$x_{n+1} = x_n - \frac{x_n^{10} - 5}{10x_n^9}$$
And so we get
$$\begin{aligned} x_1 &= 1 \\ x_2 &\approx 1.1381 \\ x_3 &\approx 1.1179 \\ x_4 &\approx 1.1161 \\ x_5 &\approx 1.1161 \end{aligned}$$
Since we have repeated the first four decimal places, we have that $\sqrt[10]{3} \approx 1.1161$.

3. Find the value for which the following equality holds:

$$\arctan(x) = x - 1.$$

Use 2 as your initial value, and approximate to the first five decimal points.

Answer:

Our function is $f(x) = \arctan(x) - x + 1$. Our initial point will be $x_1 = 2$. The derivative of the function is

$$f'(x) = \frac{1}{1+x^2} - 1.$$

Thus to compute the next approximation, we use the formula

$$x_{n+1} = x_n - \frac{\arctan(x_n) - x_n + 1}{\frac{1}{1+x_n^2} - 1}$$

And so we get

$$\begin{aligned}
x_1 &= 2 \\
x_2 &\approx 2.12857 \\
x_3 &\approx 2.13213 \\
x_4 &\approx 2.13226 \\
x_5 &\approx 2.13226
\end{aligned}$$

Since we have repeated the first five decimal places, we have that $\arctan(x) = x - 1$ when $x \approx 2.13226$.