Taylor Polynomial Practice

1. Consider the function
$$f(x) = \cos(x)$$
Compute the 5^{th} degree Taylor polynomial of $f(x)$ centered at 0.

Answer:

We need to know the first 5 derivatives of $\cos(x)$, which are
$$-\sin(x), \qquad -\cos(x), \qquad \sin(x), \qquad \cos(x), \qquad \text{and} \qquad -\sin(x),$$
respectively.

Thus the 5^{th} degree Taylor polynomial of $f(x)$ centered at 0 is
$$\begin{aligned}T_5(x) &= \frac{\cos(0)}{0!}(x-0)^0 + \frac{-\sin(0)}{1!}(x-0)^1 + \frac{-\cos(0)}{2!}(x-0)^2 \\ &+ \frac{\sin(0)}{3!}(x-0)^3 + \frac{\cos(0)}{4!}(x-0)^4 + \frac{-\sin(0)}{5!}(x-0)^5,\end{aligned}$$
which can be reduced to
$$T_5 = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

2. Consider the function
$$g(x) = \ln(x)$$
Compute the 5^{th} degree Taylor polynomial of $g(x)$ centered at 1.

Answer:

We need to know the first 5 derivatives of $\ln(x)$, which are
$$\frac{1}{x}, \qquad -\frac{1}{x^2}, \qquad \frac{2}{x^3}, \qquad -\frac{6}{x^4}, \qquad \text{and} \qquad \frac{24}{x^5},$$
respectively.

Thus the 5^{th} degree Taylor polynomial of $g(x)$ centered at 1 is
$$\begin{aligned}T_5(x) &= \frac{\ln(1)}{0!}(x-1)^0 + \frac{1}{(1)1!}(x-1)^1 + \frac{-1}{(1)^2 2!}(x-0)^2 \\ &+ \frac{2}{(1)^3 3!}(x-1)^3 + \frac{-6}{(1)^4 4!}(x-1)^4 + \frac{24}{(1)^5 5!}(x-1)^5,\end{aligned}$$
which can be reduced to
$$T_5 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$

3. Consider the function

$$h(x) = e^x$$

Compute the 5^{th} degree Taylor polynomial of $h(x)$ centered at 0. How could you use this to approximate e?

Answer:

We need to know the first 5 derivatives of e^x. But the derivative of e^x is itself. So

$$h^{(n)}(x) = \frac{d^n}{dx^n}(e^x) = e^x.$$

respectively.

Thus the 5^{th} degree Taylor polynomial of $h(x)$ centered at 0 is

$$\begin{aligned} T_5(x) = &\frac{e^0}{0!}(x-0)^0 + \frac{e^0}{1!}(x-0)^1 + \frac{e^0}{2!}(x-0)^2 \\ &+ \frac{e^0}{3!}(x-0)^3 + \frac{e^0}{4!}(x-0)^4 + \frac{e^0}{5!}(x-0)^5, \end{aligned}$$

which can be reduced to

$$T_5 = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

If we want to approximate e, we know that $h(1) = e^1 = 1$. So we know that

$$e \approx 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$