Math 1
2nd Midterm

October 20, 2016

Name (in block capital letters):

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Instructor (tick one box):

☐ **Section 1: M. Musty (10:10)**

☐ **Section 2: E. Sullivan (11:30)**

☐ **Section 3: A. Babei (12:50)**

☐ **Section 4: M. Dennis (2:10)**

Instructions: You are not allowed to provide or receive help of any kind (closed book examination). However, you may ask the instructor for clarification on problems.

1. **Wait** for **signal** to begin.
2. **Write** your **name** in the space provided, and **tick one box** to indicate which section of the course you belong to.
3. Calculators, computers, cell phones, or other computing devices are **not allowed**. In consideration of other students, please **turn off cell phones** or other electronic devices which may be disruptive.
4. Unless otherwise stated, you must **justify your solutions** to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.

Problem	Score	Possible
1		10
2		8
3		11
4		8
5		8
6		8
7		8
8		10
Total		71

1. (10 points) Which of the following statements are always true? Write "**T**" for true and "**F**" for false. Your computations will not be graded on this problem.

(a) T — $\sin(3\pi/4) = \sqrt{2}/2$.

(b) F — $\arcsin(\sqrt{2}/2) = 3\pi/4$.

(c) T — $\displaystyle\lim_{n\to\infty} \frac{5n}{12+10n} = \frac{1}{2}$.

(d) F — $\displaystyle\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 0$.

(e) F — Rational functions are continuous on $(-\infty, \infty)$.

(f) F — If $\displaystyle\lim_{x\to a^+} f(x) = L$, then $\displaystyle\lim_{x\to a^-} f(x) = L$.

(g) T — If $\displaystyle\lim_{x\to a} f(x) = L$, then $\displaystyle\lim_{x\to a^-} f(x) = L$.

(h) F — The function $f(x) = \frac{(x-1)(x-2)}{x-1}$ has an infinite discontinuity at 1.

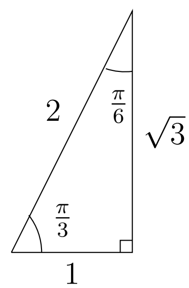
(i) F — A function can have at most one horizontal asymptote.

(j) F — If a is not in the domain of f, then $\displaystyle\lim_{x\to a} f(x)$ does not exist.

2. (8 points) Find the following values. Show your work.

(a) $\cos\left(\frac{\pi}{6}\right)$

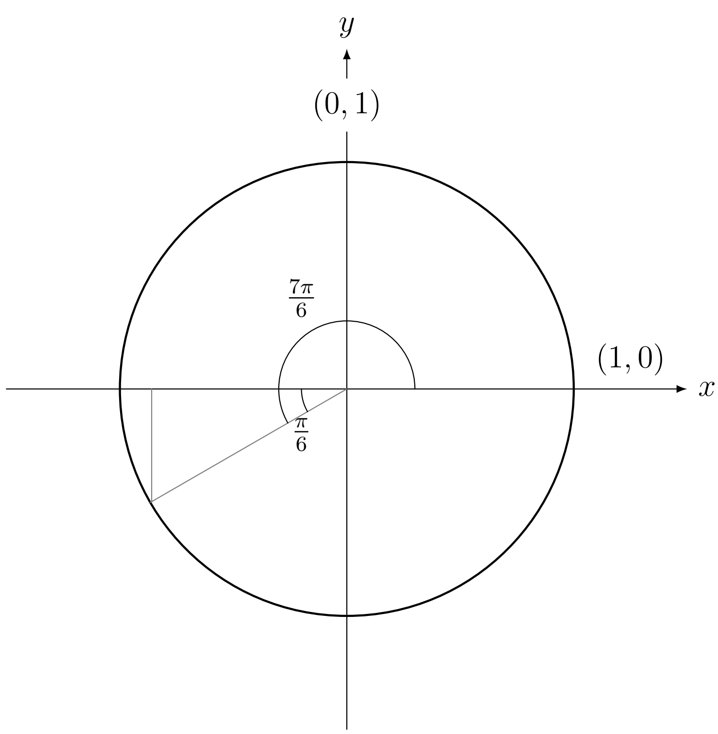
Solution:
We look at the following triangle, which we obtain by splitting an equilateral triangle of side length 2 in half:



Then $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

(b) $\sin\left(\frac{7\pi}{6}\right)$

Solution:
Since $\pi < \frac{7\pi}{6} < \frac{9\pi}{6} = \frac{3\pi}{2}$, the angle $\frac{7\pi}{6}$ is in the third quadrant, so the value of $\sin(\frac{7\pi}{6})$ given by the y-coordinate of the angle will be negative.



The part of the angle in the third quadrant is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$. Thus, the y-coordinate of the point representing the angle $\frac{7\pi}{6}$ is given by the negative of the opposite side of the angle $\frac{\pi}{6}$ in the triangle depicted above. This is $-\sin(\frac{\pi}{6}) = -\frac{1}{2}$.

(c) $\arcsin(\sin(\pi))$

Solution:
We cannot use the cancellation laws since π is not in the range $[-\pi/2, \pi/2]$ of $\arcsin(x)$. Instead,

$$\arcsin(\sin(\pi)) = \arcsin(0) = 0,$$

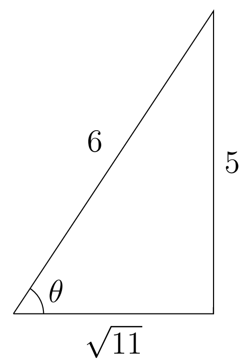
since $\sin(\pi) = 0$ and the only angle θ in the interval $[-\pi/2, \pi/2]$ with $\sin(\theta) = 0$ is $\theta = 0$.

3. (11 points) For each of the following inverse trigonometry problems, draw the corresponding triangles, and evaluate the expression. Show all your work.

(a)

$$\cos\left(\arcsin\left(\frac{5}{6}\right)\right)$$

Solution:



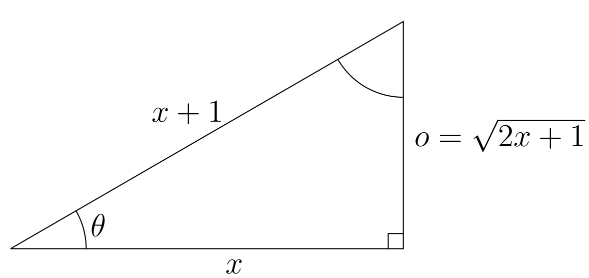
So

$$\cos\left(\arcsin\left(\frac{5}{6}\right)\right) = \frac{\sqrt{11}}{6}$$

(b) Simplify the following expression so that in has no trigonometric functions.

$$\tan\left(\arccos\left(\frac{x}{x+1}\right)\right)$$

Solution:



Let o denote the opposite side of the triangle. Then

$$x^2 + o^2 = (x+1)^2,$$

and so

$$o^2 = (x+1)^2 - x^2.$$

Thus

$$\begin{aligned} o &= \sqrt{(x+1)^2 - x^2} \\ &= \sqrt{(x^2 + 2x + 1) - x^2} \\ &= \sqrt{2x+1} \end{aligned}$$

So

$$\tan\left(\arccos\left(\frac{x}{\sqrt{x+1}}\right)\right) = \frac{\sqrt{2x+1}}{x}$$

4. (8 points) Evaluate the limit, if it exists. Show all your work.

   (a)

$$\lim_{x\to 2}\frac{x^2+x-6}{(x-2)}$$

Solution: We factor the top to get

$$\lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)}.$$

Now

$$x+3=\frac{(x-2)(x+3)}{(x-2)}$$

for all x except 2. Thus

$$\lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)}=\lim_{x\to 2}(x+3)=5.$$

   (b)

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4x-x^2}$$

Solution:

$$\frac{2-\sqrt{x}}{4x-x^2}=\frac{2-\sqrt{x}}{4x-x^2}\cdot\frac{2+\sqrt{x}}{2+\sqrt{x}}=\frac{4-x}{(4x-x^2)(2+\sqrt{x})}=\frac{4-x}{x(4-x)(2+\sqrt{x})}$$

Since

$$\frac{4-x}{x(4-x)(2+\sqrt{x})}=\frac{1}{x(2+\sqrt{x})}$$

for all x except 4, we have

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4x-x^2}=\lim_{x\to 4}\frac{1}{x(2+\sqrt{x})}=\frac{1}{4(2+\sqrt{4})}=\frac{1}{4(2+2)}=\frac{1}{4(4)}=\frac{1}{16}$$

5. (8 points)

(a) Let f be a function such that $2x+3 \leq f(x) \leq \left(\frac{x}{3}+2\right)^2$ when $0 \leq x \leq 5$. Evaluate $\lim_{x\to 3} f(x)$.

When I take the limits of each side, I get that

$$\lim_{x\to 3} 2x+3 = 9 = \lim_{x\to 3}\left(\frac{x}{3}+2\right)^2.$$

Thus by the Squeeze Theorem, we have that $\lim_{x\to 3} f(x) = 9$.

(b) Evaluate $\lim_{x\to 1}(x-1)^2 \sin\left(\frac{1}{1-x}\right)$.

First note that $-1 \leq \sin\left(\frac{1}{1-x}\right) \leq 1$. If we multiply all sides by $(x-1)^2$, we get $-(x-1)^2 \leq (x-1)^2 \sin\left(\frac{1}{1-x}\right) \leq (x-1)^2$. But when I take the limits of each side, I get

$$\lim_{x\to 1} -(x-1)^2 = 0 = \lim_{x\to 1}(x-1)^2.$$

Thus by the Squeeze Theorem, we have $\lim_{x\to 1}(x-1)^2 \sin\left(\frac{1}{1-x}\right) = 0$.

6. (8 points) For the following functions, find all vertical and horizontal asymptotes. If there are no vertical or horizontal asymptotes, right NONE.

(a) $f(x) = \frac{x^2-4}{(2x+3)(x-1)}$.

Horizontal Asymptotes: There is a horizontal asymptote at $y = \frac{1}{2}$.

Vertical Asymptotes: There are vertical asymptotes at $x = -\frac{3}{2}$ and $x = 1$.

(b) $g(x) = \frac{x^2-9}{x+3}$.

Horizontal Asymptotes: NONE

Vertical Asymptotes: NONE

(c) $h(x) = \arctan(4x)$.

Horizontal Asymptotes: There are horizontal asymptotes at $\frac{-\pi}{2}$ and $\frac{\pi}{2}$.

Vertical Asymptotes: NONE

7. (8 points) For each of the sequences below, determine if the sequence converges or not. If it converges, find the limit. Justify your answers.

(a) $a_n = \dfrac{(n-1)(n^2+1)}{(3n-1)(2n+5)}$

Solution:
Expading the factors, have that

$$a_n = \frac{(n-1)(n^2+1)}{(3n-1)(2n+5)} = \frac{n^3+n-n^2-1}{6n^2+15n-2n-5} = \frac{n^3-n^2+n-1}{6n^2+13n-5}.$$

Dividing both numerator and denominator by n^2 (since we are looking for limits as $n \to \infty$ where n is large, we do not worry about $n = 0$) we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1+\frac{1}{n}-\frac{1}{n^2}}{6+\frac{13}{n}-\frac{5}{n^2}} = \lim_{n\to\infty} \frac{n-1}{6} = \infty.$$

The second equality follows since in the long run $\frac{1}{n} \to 0$, $\frac{1}{n^2} \to 0$, $\frac{13}{n} \to 0$ and $\frac{5}{n^2} \to 0$. The third equality follows since in the long run the numerator will grow much larger than the denominator.

Thus, the sequence does not converge.

(b) $b_n = e^{-(n^2)}$

Solution:
Since

$$b_n = e^{-n^2} = \left(\frac{1}{e}\right)^{n^2},$$

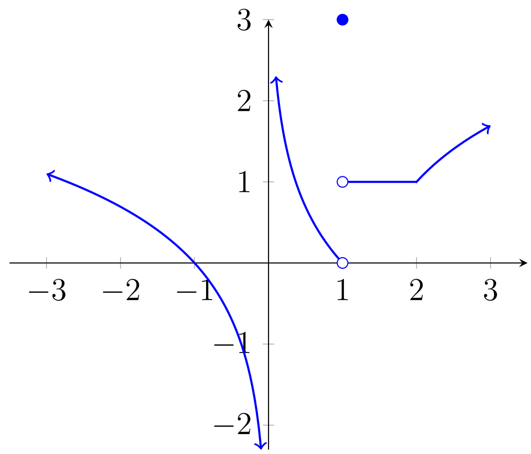
$\left(\frac{1}{e}\right)^{n^2}$ gets very small as n gets very large, and $\lim\limits_{n\to\infty} b_n = 0$.

(c) $c_n = \dfrac{(-2)^n}{3^n}$

Solution:
Since $c_n = \frac{(-2)^n}{3^n} = \left(\frac{-2}{3}\right)^n$, and $-1 < -\frac{2}{3} \leq 1$, the sequence converges to 0 and

$$\lim_{n\to\infty} c_n = 0.$$

8. (10 points) Let f be defined by the graph below.



(a) Compute $\lim_{x \to 0^-} f(x)$.
Solution:

$$\lim_{x \to 0^-} f(x) = -\infty.$$

(b) Find the interval(s) where f is continuous.
Solution:
f is continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

(c) Find the discontinuities of f. For each discontinuity of f, determine its type (removable, jump, or infinite).
Solution:
f has an infinite discontinuity at 0 and a jump discontinuity at 1.

(d) Compute

$$\lim_{x \to \infty} f\left(\frac{2x^2+5}{x^2+1}\right).$$

Solution:
Since f is continuous at $\lim_{x \to \infty} \frac{2x^2+5}{x^2+1} = 2$, we get that

$$\begin{aligned} \lim_{x \to \infty} f\left(\frac{2x^2+5}{x^2+1}\right) &= f\left(\lim_{x \to \infty} \frac{2x^2+5}{x^2+1}\right) \\ &= f(2) = 1. \end{aligned}$$