Practice problems review I

Exercise 1: domains and ranges; inverse functions

(1) Let $f(x) = \sin(x), g(x) = \arcsin(x)$. What are the domains and ranges of $f, g, f \circ g$ and $g \circ f$?

Solution: The domain for $f(x) = \sin(x)$ is $(-\infty, +\infty)$, the range is $[-1, 1]$. The domain for $g(x) = \arcsin(x)$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. Since $(f \circ g)(x) = \sin(\arcsin(x))$, the domain of $f \circ g$ is contained in the domain of $\arcsin(x)$, and since the domain of $\sin(x)$ are all the real numbers, we have no further restriction. So the domain of $f \circ g$ is $[-1, 1]$. Since $(f \circ g)(x) = \sin(\arcsin(x)) = x$ when $-1 \leq x \leq 1$, the range is $[-1, 1]$.

Since $(g \circ f)(x) = \arcsin(\sin(x))$, the domain of $g \circ f$ is contained in the domain of $\sin(x)$, as long as the range of $\sin(x)$ is in the domain of $\arcsin(x)$. Since $\sin(x)$ has domain all real numbers, and the range of $\sin(x)$ is $[-1, 1]$ and contained in the domain of $\arcsin(x)$, the domain of $g \circ f$ is all of the domain os $\sin(x)$, that is $(-\infty, +\infty)$. The range of $g \circ f$ is the whole range of $\arcsin(x)$, since the range of $\sin(x)$ is the hole domain of $\arcsin(x)$ (if this is still confusing, it might be helpful to draw a picture of the two compositions and track domains and ranges)

(2) Let $f(x) = x^2, g(x) = \sqrt{x+1}$. What are the domains and ranges of $f, g, f \circ g$ and $g \circ f$?

Solution: The domain of $f(x) = x^2$ is $(-\infty, +\infty)$, the range is $[0, \infty)$. The domain of $g(x) = \sqrt{x+1}$ is $[-1, +\infty)$, the range is $[0, +\infty)$ (this is just the function $\sqrt{x}$ moved horizontally 1 to the left, which doesn't affect the range.) $f \circ g(x) = (\sqrt{x+1})^2$, so the domain is $[-1, +\infty)$. Due to this, $(f \circ g)(x) = (\sqrt{x+1})^2 = x+1$ when $x \geq -1$, so the range is $[0, +\infty)$.

$(g \circ f)(x) = \sqrt{x^2+1}$, and since $x^2+1 \geq 0$ always, the domain of $g \circ f$ is $(-\infty, +\infty)$. Since $x^2 + 1 \geq 1$, then $\sqrt{x^2+1} \geq \sqrt{1}$, so the range of $g \circ f$ is $[1, +\infty)$.

(3) Let $f(x) = x + 2, g(x) = \sqrt{x+2}$. What are the domains and ranges of $\frac{f}{g}$ and $\frac{g}{f}$?

Solution: $\frac{f(x)}{g(x)} = \frac{x+2}{\sqrt{x+2}}$, so $\sqrt{x+2} \neq 0$ and $x + 2 \geq 0$, so the domain is $(-2, +\infty)$. Given this domain, $\frac{f(x)}{g(x)} = \frac{x+2}{\sqrt{x+2}} = \sqrt{x+2}$ when $x > -2$, so $x + 2 > 0$ and $\sqrt{x+2} > \sqrt{0}$. Therefore the range is $(0, +\infty)$.

(4) Let $f(x) = e^x, g(x) = \ln(x+3)$. What are the domains and ranges of $f \circ g$ and $g \circ f$?

Solution: The domain of $f(x)$ is $(-\infty, +\infty)$, the range is $(0, +\infty)$. The Domain od $g(x)$ is $(-3, +\infty)$, the range is $(-\infty, +\infty)$ (this is just the function $\ln(x)$ moved 3 to the left, so the range doesn't change).

$(f \circ g)(x) = e^{\ln(x+3)}$, so the domain is $(-3, +\infty)$. Given this domain, $(f \circ g)(x) = e^{\ln(x+3)} = x + 3$ when $x > -3$, so the range is $(0, +\infty)$

$(g \circ f)(x) = \ln(e^x + 3)$, so the domain is $(-\infty, +\infty)$ because $e^x + 3 > 3$for any x. Then $\ln(e^x + 3) \geq \ln(3)$ so the range is $(\ln(3), +\infty)$.

Exercise 2: logarithm and exponential equations. Solve for x:

(1) $\ln(x+2)+\ln(x-2)=\ln(6)$

Solution: First, the equation makes sense when both $x+2>0$ and $x-2>0$, so $x>2$. Then

$$\ln(x+2)+\ln(x-2)=\ln(6)$$
$$\ln[(x+2)(x-2)]=\ln(6)$$
$$\ln(x^2-4)=\ln(6)$$

Raising e to the both sides, we have

$$e^{\ln(x^2-4)}=e^{\ln(6)}$$
$$x^2-4=6$$
$$x^2=10$$
$$x=\pm\sqrt{10}.$$

Since $-\sqrt{10}\approx-3.16<-2$, it doesn't satisfy the conditions we have started with, so the only solution is $x=\sqrt{10}$.

(2) $\ln(x+3)-\ln(x-3)=\ln(5)$

Solution: First, the equation makes sense when both $x+3>0$ and $x-3>0$, so $x>3$. Then

$$\ln(x+3)-\ln(x-3)=\ln(5)$$
$$\ln\frac{x+3}{x-3}=\ln(5)$$

Raising e to the both sides, we have

$$e^{\ln\frac{x+3}{x-3}}=e^{\ln(5)}$$
$$\frac{x+3}{x-3}=5$$
$$x+3=5(x-3)$$
$$x+3=5x-15$$
$$5x-x=3+15$$
$$4x=18$$
$$x=4.5$$

Since $4.5>3$, $x=4.5$ is a solution.

(3) $2^{3x+1}=4^x$

Solution: The equation is defined for any real x. We can take $\log_2$ of both sides:

$$\log_2(2^{3x+1})=\log_2(4^x)$$
$$3x+1=x\log_2(4)$$
$$3x+1=2x$$
$$x=-1$$

.

(4) $2^{e^x} = e^{2^x}$

Solution: The equation is defined for any real x. We can take $\log_2$ of both sides again:

$$\log_2(2^{e^x}) = \log_2(e^{2^x})$$
$$e^x = 2^x \log_2(e)$$

Now we can take ln of both sides:

$$\ln(e^x) = \ln(2^x \log_2(e))$$
$$x = \ln(2^x) + \ln(\log_2(e))$$
$$x = x\ln(2) + \ln(\log_2(e))$$
$$x - x\ln(2) = \ln(\log_2(e))$$
$$x(1 - \ln(2)) = \ln(\log_2(e))$$
$$x = \frac{\ln(\log_2(e))}{1 - \ln(2)}$$

is our solution, since $1 - \ln(2) \neq 0$.

Exercise 3: library of functions. Consider the following classes of functions: linear, power, polynomial, rational, algebraic. For each of the following functions, write down which classes it belongs to and which classes it doesn't belong to (all five classes should be mentioned).

(1) $f(x) = 1$ is linear, power ($1 = x^0$), polynomial, rational and algebraic (since it is already linear)

(2) $g(x) = x^2 + 1$ is polynomial, rational and algebraic. It is not power or linear.

(3) $h(x) = \sqrt{x^3}$ is algebraic, power ($\sqrt{x^3} = x^{3/2}$). It is not linear, polynomial or rational.

(4) $k(x) = \dfrac{x+1}{x+1}$ is rational and algebraic. It is not linear, power or polynomial since it is not defined at $x = -1$.

(5) $f(x) = x^{3\pi}$ is power. It is not linear, polynomial, rational or algebraic.

Exercise 4: Let $f(x)$ be a function with domain $[-2, 3]$ and range $[0, 8]$. What are the domains and ranges of the following functions?

(1) $-f(-x-1)$

Solution:

$$-2 \leq -x - 1 \leq 3,$$

so

$$-1 \leq -x \leq 4$$
$$-4 \leq x \leq 1,$$

and therefore the domain is $[-4, 1]$. Since

$$0 \leq f(-x-1) \leq 8,$$
$$-8 \leq -f(-x-1) \leq 0,$$

so the range is $[-8, 0]$.

(2) $3f(2x+1)$

Solution:

$$-2 \leq 2x+1 \leq 3$$

so

$$-3 \leq 2x \leq 2$$
$$-1.5 \leq x \leq 1,$$

and therefore the domain is $[-1.5, 1]$. Since

$$0 \leq f(2x+1) \leq 8,$$
$$0 \leq 3f(2x+1) \leq 24,$$

so the range is $[0, 24]$.

(3) $4f^{-1}(-x)+1$

Solution:

We are dealing with f^{-1} not f. Since f^{-1} has domain $[0, 8]$ and range $[-2, 3]$,

$$0 \leq -x \leq 8$$
$$-8 \leq x \leq 0$$

so the domain is $[-8, 0]$. Since

$$-2 \leq f^{-1}(-x) \leq 3$$
$$-8 \leq 4f^{-1}(-x) \leq 12$$
$$-7 \leq 4f^{-1}(-x)+1 \leq 13$$

so the range is $[-7, 13]$

Exercise 5: True/False Are the following statements true or false?

(1) $\sin(x)$ is an even function: false (look at the graph)
(2) $\sin(x)$ is an odd function: true(look at the graph)
(3) $\cos(x)$ is an even function: true
(4) $\cos(x)$ is an odd function: false
(5) e^x is an increasing function: true
(6) $\ln(x)$ is a decreasing function: false
(7) The sequence $a_n = \frac{2n+1}{3n}$ is bounded by 2/3: false
(8) The function $\frac{3x^2}{5x-1}$ is even : false
(9) The function $(x-5)^2+5$ is one-to-one on the interval $[-1, 5]$: true
(10) The function $(x-5)^2+5$ is one-to-one on the interval $[0, 7]$: false , plugging in $x = 6$ and $x = 4$ gives the same number