MATH 1: INTRODUCTION TO CALCULUS
MIDTERM EXAM #1 SOLUTIONS

Problem 1. [12 points]

(a) Simplify $2\ln(e^3)$.

$$2\ln(e^3) = 2(3) = \boxed{6}$$

(b) Let $f(x) = \cos(x)$ and $g(x) = e^{x^2}$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

$$(g \circ f)(x) = g(\cos(x)) = e^{\cos(x)^2} = \boxed{e^{\cos^2(x)}}$$

$$(f \circ g)(x) = f(e^{x^2}) = \boxed{\cos(e^{x^2})}$$

(c) Solve the equation $\ln(\sqrt{x-5}) = 1$ for x.

$$\ln(\sqrt{x-5}) = 1$$
$$\sqrt{x-5} = e^1$$
$$x - 5 = e^2$$
$$\boxed{x = e^2 + 5}$$

(d) Solve the equation $9^{(3^x)} = 3^{(27^x)}$ for x.

$$9^{(3^x)} = 3^{(27^x)}$$
$$3^{(2*3^x)} = 3^{27^x}$$
$$2 \cdot 3^x = 27^x$$
$$2 \cdot 3^x = 3^{3x}$$
$$\log_3 2 + x = 3x$$
$$\log_3 2 = 2x$$
$$\boxed{x = \frac{1}{2}\log_3 2}$$

Date: Thursday, 10 October 2019.

Problem 2. [9 points]

(a) Find all solutions of $2\cos^2(\theta)\tan(\theta) = \sin(\theta)$ for θ in the interval $[0, 2\pi]$.

$$2\cos^2(\theta)\tan(\theta) = \sin(\theta)$$
$$2\cos^2(\theta)\frac{\sin(\theta)}{\cos(\theta)} = \sin(\theta)$$
$$2\cos(\theta)\sin(\theta) = \sin(\theta)$$
$$2\cos(\theta)\sin(\theta) - \sin(\theta) = 0$$
$$\sin(\theta)(2\cos(\theta) - 1) = 0$$
$$\sin(\theta) = 0 \text{ or } 2\cos(\theta) - 1 = 0$$
$$\sin(\theta) = 0 \text{ or } \cos(\theta) = \frac{1}{2}$$
$$\boxed{\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}}$$

(b) Let $f(x) = (x-3)^2$ for $x \leq 3$. Find $f^{-1}(x)$.

$$y = (x-3)^2$$
$$y = (x-3)^2$$
$$\sqrt{y} = \pm(x-3)$$
$$3 \pm \sqrt{y} = x$$
$$\boxed{f^{-1}(x) = 3 - \sqrt{x}}$$

(c) Solve the equation $\log_2(\sqrt{x}) + \log_2(\sqrt[3]{x}) = 2$ for x.

$$\log_2(\sqrt{x}) + \log_2(\sqrt[3]{x}) = 2$$
$$\log_2(\sqrt{x} \cdot \sqrt[3]{x}) = 2$$
$$\sqrt{x} \cdot \sqrt[3]{x} = 2^2$$
$$x^{\frac{5}{6}} = 4$$
$$\boxed{x = 4^{\frac{6}{5}}}$$

Problem 3. [8 points] Mark the following statements as true or false.

False $\ln(e^{-2}) = e^{\ln(-2)}$

False If k is positive, 10^{-k} is negative.

False $\log_5(xy) = \log_5(x)\log_5(y)$

True If a is a positive, constant, then $\ln(a^r) = r\ln(a)$.

False $\log_3 0 = 1$

False The domain of $f(x) = 2^x + 1$ is $(2, \infty)$.

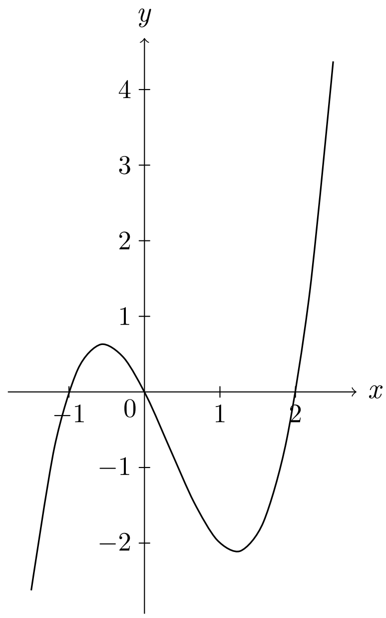
False The graph of $y = e^{\ln x}$ is a parabola.

True The slope of a secant line passing through the point $(a, f(a))$ and a nearby point on the graph of $f(x)$ approximates the instantaneous rate of change of $f(x)$ at a.

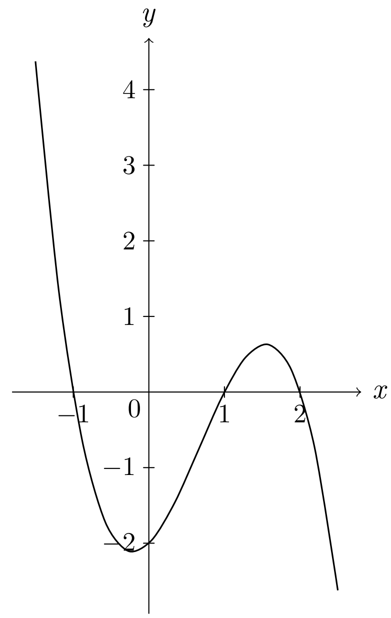
Problem 4. [7 points]

(1) Match the transformations of $f(x)$ with their graphs.

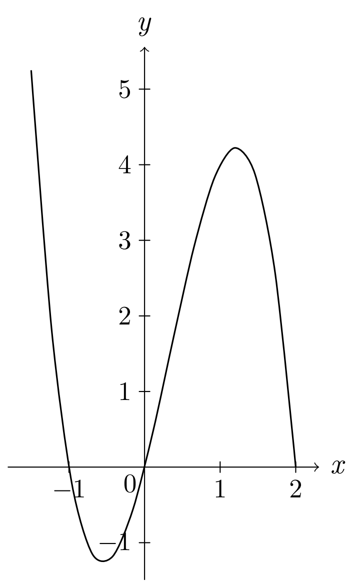
Options: $-2f(x)$, $f(x/2)$, $f(1-x)$



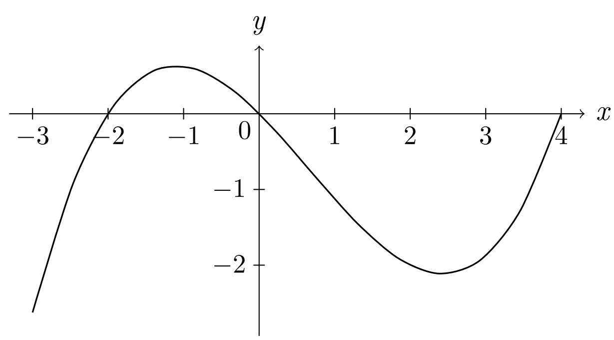
$f(x)$



$f(1-x)$



$-2f(x)$



$f(x/2)$

(2) Is $f(x)$ even, odd, or neither?

neither

Problem 5. [12 points]

(a) Plot the graph of $f(x) = e^x$ and its inverse function $f^{-1}(x) = \ln(x)$.

(b) Let $g(x) = e^{2x}$ and $h(x) = e^{x+1}$. What are $g^{-1}(x)$ and $h^{-1}(x)$?

$$g^{-1}(x) = \frac{1}{2}\ln x$$
$$h^{-1}(x) = \ln x - 1$$

(c) Describe which **basic** transformation we need to

- transform the graph of $f(x) = e^x$ into the graph of $g(x) = e^{2x}$.
  horizontal compression by a factor of 2

- transform the graph of $f^{-1}(x) = \ln(x)$ into the graph of $g^{-1}(x)$, as found above.
  vertical compression by a factor of 2

(d) Describe which **basic** transformation we need to

- transform the graph of $f(x) = e^x$ into the graph of $h(x) = e^{x+1}$.
  shift left by 1

- transform the graph of $f^{-1}(x) = \ln(x)$ into the graph of $h^{-1}(x)$ as found above.
  shift down by 1

(e) Fill in the blanks below with options from the bank of terms.

If we horizontally compress the original function by a factor of a, its corresponding inverse function will be vertically compressed by a factor of a ($a > 1$).
(There are other correct answers)

Term Bank: horizontally compress, horizontally stretch, vertically compress, vertically stretch

If we horizonally shift left the original function a ($a > 0$) units, its corresponding inverse function will be vertically shifted down a units ($a > 0$).
(There are other correct answers)

Term Bank: horizontally shift left, horizontally shift right, vertically shift up, vertically shift down

Problem 6. [8 points]

The number of hours of daylight in a northeastern city is modeled by the function

$$N(t) = 12 + 3\sin[\frac{2\pi}{365}(t-79)]$$

where t is the number of days after January 1.

(a) Find the amplitude and period of $N(t)$.

The amplitude is 3, and the period is 365.

(b) How many hours of sunlight does the model predict on the longest day of the year?

The sine function fluctuates between -1 and 1, so the maximum value of $\sin(x)$ is 1. Therefore, the maximum value of $N(t)$ is $12 + 3(1) = 15$. On the longest day of the year, the model predicts 15 hours of sunlight.

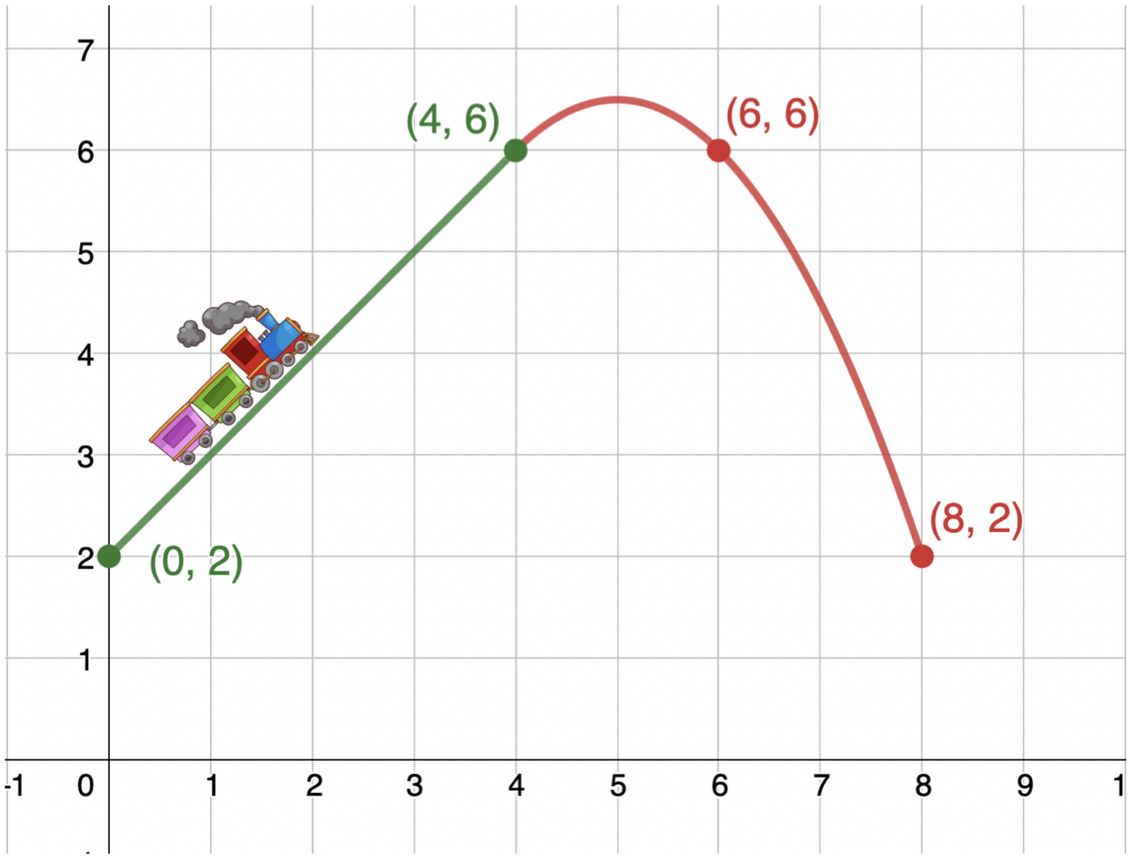
(c) How many hours of sunlight does the model predict 90 days after January 1?

$$N(90) = 12 + 3\sin[\frac{2\pi}{365}(90-79)] = 12 + 3\sin[\frac{2\pi}{365}\cdot 11)]$$

$$= \boxed{12 + 3\sin[\frac{22\pi}{365}]}$$

Problem 7. [9 points]

The figure below shows a track of the Green-Red Mountain. The first part of the mountain is linear, while the rest is part of the quadratic curve $-\frac{1}{2}x^2 + 5x - 6$.



(a) Suppose the figure depicts the entire mountain, modeled by a function $f(x)$. Using the given coordinates and the formula of the quadratic curve, write down a piecewise definition of $f(x)$.

$$f(x) = \begin{cases} x + 2 & x \leq 4 \\ -\frac{1}{2}x^2 + 5x - 6 & x \geq 4 \end{cases}$$

(b) Engineers are working to decide if it is safe for a train to travel on the mountain or if they need to build a tunnel through the mountain. If the absolute value of the slope at the foot of the mountain, which is the point $(8, 2)$, is greater than 2, it is too dangerous to run a train, and they need to build a tunnel. Compute the slope of the secant line through $(7, \frac{9}{2})$ and $(8, 2)$, and the slope of the secant line through $(\frac{15}{2}, \frac{27}{8})$ and $(8, 2)$. Use these slopes to estimate the slope of the tangent line at $(8, 2)$ and help the engineers make the decision.

$$m_1 = \frac{2 - \frac{9}{2}}{8 - 7} = \frac{-\frac{5}{2}}{1} = -\frac{5}{2}$$
$$m_2 = \frac{2 - \frac{27}{8}}{8 - \frac{15}{2}} = \frac{-\frac{11}{8}}{\frac{1}{2}} = -\frac{22}{8} = -\frac{11}{4}$$

Based on these two slopes, we estimate that the slope of the tangent line at $(8, 2)$ is around -3, or at least steeper than -2, and thus not safe for the train.

Problem 8. [6 points]

Show algebraically whether the following functions are even, odd, or neither:

(a) $f(x) = e^{|x|}$

$$f(-x) = e^{|-x|} = e^{|x|} = f(x)$$

Therefore, f must be $\boxed{\text{even}}$.

(b) $g(x) = \sin x \cos x$

$$g(-x) = \sin(-x)\cos(-x) = -\sin(x)\cos(x) = -g(x)$$

Therefore, g must be $\boxed{\text{odd}}$.

(c) $h(x) = x \sin x$

$$h(-x) = (-x)\sin(-x) = (-x) \cdot (-\sin(x)) = x\sin(x) = h(x)$$

Therefore, h must be $\boxed{\text{even}}$.