MATH 1: INTRODUCTION TO CALCULUS
MIDTERM EXAM #2

Name: ______________________________

Section (please circle): 10 Winkeler 11 Tripp 12 Chen 2 Xiao

(1) Write your name *legibly* and circle your section above.
(2) This exam is closed book. You may not use notes, calculators, or any other external resource. It is a violation of the honor code to give or receive help on this exam.
(3) You must explain what you are doing, justify your answer, and show your work. You will be *graded on your work*, not just on your answer. Please write clearly.
(4) It is fine to leave your answer in a form such as $14\sqrt{239}$. However, if an expression can be easily simplified (such as $\cos(\pi/2)$ or $2+3$), please simplify it.
(5) If you use the backside of a page or the scratch paper at the end of the exam, and you want us to look at it, please write on a problem "Continued on back/page…".

Problem	**Score**	**Problem**	**Score**
1	/10	6	/6
2	/8	7	/12
3	/8	8	/8
4	/10		
5	/12		
		Total	/74

Date: Thursday, 31 October 2019.

Problem 1.[10 points] Mark the following statements as true or false.

True — $f'(a)$ is the slope of the tangent line to $f(x)$ at a.

True — $g(x) = \frac{1}{x}$ is continuous on its domain.

False — $2, 4, 6, 8, 10, 12, \ldots$ is a geometric sequence.

True — The sequence $a_n = \sin(n)$ is bounded.

False — $\lim_{x \to 0} \ln x = -\infty$.

False — If $h(x) = 2x$, then $h'(x) = x^2$.

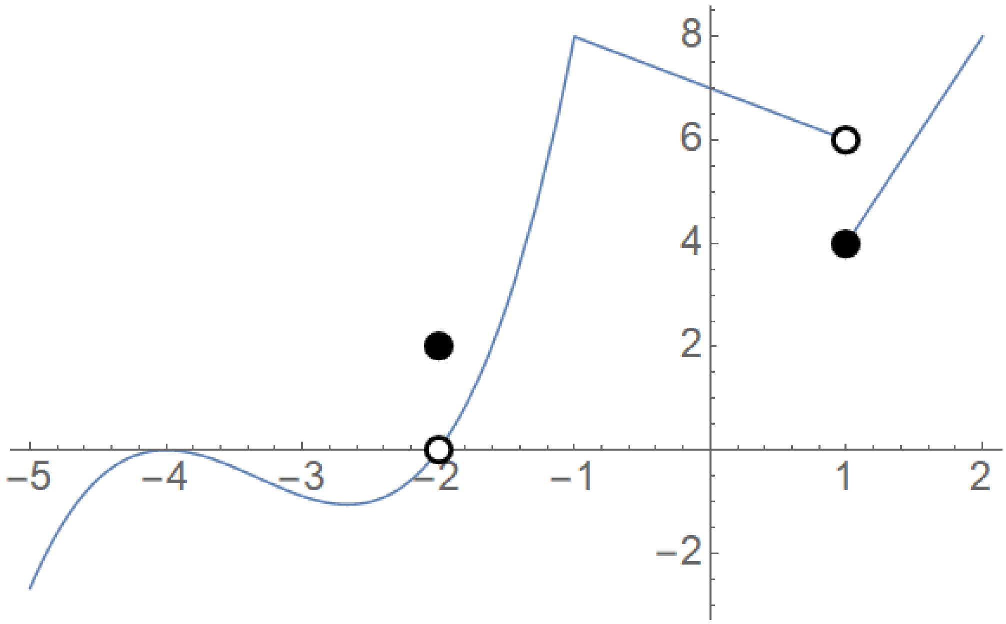
False — If $j(x)$ is a continuous function such that $j(0) = 0$ and $j(1) = 1$, then $j(\frac{1}{2}) = \frac{1}{2}$.

True — A monotonic sequence is either bounded above or bounded below.

False — The value of a continuous function cannot approach ∞ as $x \to \infty$.

False — The derivative of $g(x) = |x|$ at $x = 0$ is 0.

Problem 2. [8 points] Use the graph of $y = f(x)$ below to answer the following questions.



(a) Find the following limits. Write DNE if the limit does not exist.

(i) $\lim_{x\to -4^+} f(x) = \underline{0}$

(ii) $\lim_{x\to -2} f(x) = \underline{0}$

(iii) $\lim_{x\to -1} f(x) = \underline{8}$

(iv) $\lim_{x\to 1^-} f(x) = \underline{6}$

(v) $\lim_{x\to 1} f(x) = \underline{\text{DNE}}$

(b) At what values of x is $f(x)$ not differentiable?

$x = -2, -1, 1$

Problem 3. [8 points]

(a) Use the Squeeze Theorem to find $\lim_{x\to 0} x^2 \sin(\ln|x|)$

$$-x^2 \le x^2 \sin(\ln|x|) \le x^2$$
$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin(\ln|x|) \le \lim_{x\to 0} x^2$$
$$0 \le \lim_{x\to 0} x^2 \sin(\ln|x|) \le 0$$

So $\lim_{x\to 0} x^2 \sin(\ln|x|) = \boxed{0}$.

(b) Use the Intermediate Value Theorem to show that $f(x) = x + 2^x$ has a zero.

Since $f(-1) = -1 + 2^{-1} = -\frac{1}{2}$ and $f(1) = 1 + 2^1 = 3$, $f(x)$ must have a zero somewhere on the interval $[-1, 1]$.

Problem 4. [10 points]

(a) State the three requirements for $f(x)$ to be continuous at a point a.

(a) $f(a)$ is defined.

(b) $\lim_{x \to a} f(x)$ exists.

(c) $\lim_{x \to a} f(x) = f(a)$.

(b) List the three types of discontinuities a function can have.

removable, jump, infinite

(c) Find and classify all discontinuities of $f(x) = \dfrac{x+1}{x^2-1}$.

$f(x)$ has a removable discontinuity at $x = -1$ and an infinite discontinuity at $x = 1$.

Problem 5. [12 points] Consider the piecewise function

$$f(x) = \begin{cases} x+2, & x < 2 \\ a, & x = 2 \\ \frac{c(x^2-b^2)}{x-2}, & x > 2 \end{cases}.$$

where a, b and c are some constants.

(a) Let $c = 1$. Solve for a and b so that this function is continuous everywhere.

First, we calculate $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+2 = 2+2 = 4$. In order for $f(x)$ to be continuous at 2, it must be that $f(2) = a = 4$ as well then. Finally, we want to make sure that $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{(x^2-b^2)}{x-2} = 4$ as well. Letting $b = 2$ lets us cancel $x-2$ from the numerator and denominator, giving us $\lim_{x\to 2^+} \frac{(x^2-2^2)}{x-2} = \lim_{x\to 2^+} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2^+} x+2 = 4$.

Therefore, the answer is $\boxed{a = 4, b = 2}$. Alternatively, the answer $\boxed{a = 4, b = -2}$ also works.

(b) Let $a = 4$, $b = 3$, $c = 1$, so we get

$$f(x) = \begin{cases} x+2, & x < 2 \\ 4, & x = 2 \\ \frac{(x^2-9)}{x-2}, & x > 2 \end{cases}.$$

Is $f(x)$ continuous at $x = 2$? If not, what type of discontinuity does it have?

Since $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{(x^2-9)}{x-2} = -\infty$, $f(x)$ has an $\boxed{\text{infinite discontinuity at } x = 2}$.

(c) Let $a = 3$, $b = 2$, $c = 1$, so we get

$$f(x) = \begin{cases} x+2, & x < 2 \\ 3, & x = 2 \\ \frac{(x^2-4)}{x-2}, & x > 2 \end{cases}.$$

Is $f(x)$ continuous at $x = 2$? If not, what type of discontinuity does it have?

Since $\lim_{x\to 2} f(x) = 4$ but $f(2) = 3$, $f(x)$ has a $\boxed{\text{removable discontinuity at } x = 2}$.

(d) Let $a = 1$, $b = 2$, $c = 3$, so we get

$$f(x) = \begin{cases} x+2, & x < 2 \\ 1, & x = 2 \\ \frac{3(x^2-4)}{x-2}, & x > 2 \end{cases}.$$

Is $f(x)$ continuous at $x = 2$? If not, what type of discontinuity does it have?

Since $\lim_{x\to 2^-} f(x) = 4$ but $\lim_{x\to 2^+} f(x) = 12$, $f(x)$ has a $\boxed{\text{jump discontinuity at } x = 2}$.

Problem 6. [6 points]

(a) Suppose $\lim_{x\to 2} f(x) = 3$. Use the algebraic limit laws to compute $\lim_{x\to 2} \sqrt{10-(f(x))^2}$.

$$\begin{aligned}
\lim_{x\to 2} \sqrt{10-(f(x))^2} &= \sqrt{\lim_{x\to 2}(10-(f(x))^2)} \\
&= \sqrt{\lim_{x\to 2} 10 - \lim_{x\to 2}(f(x))^2} \\
&= \sqrt{10-(\lim_{x\to 2} f(x))^2} \\
&= \sqrt{10-(3)^2} \\
&= \sqrt{1} \\
&= \boxed{1}
\end{aligned}$$

(b) Suppose $\lim_{x\to 3} g(x) = 4$ and $\lim_{x\to 3} h(x) = 1$. Use the algebraic limit laws to compute $\lim_{x\to 3} \frac{4g(x)}{1+h(x)}$.

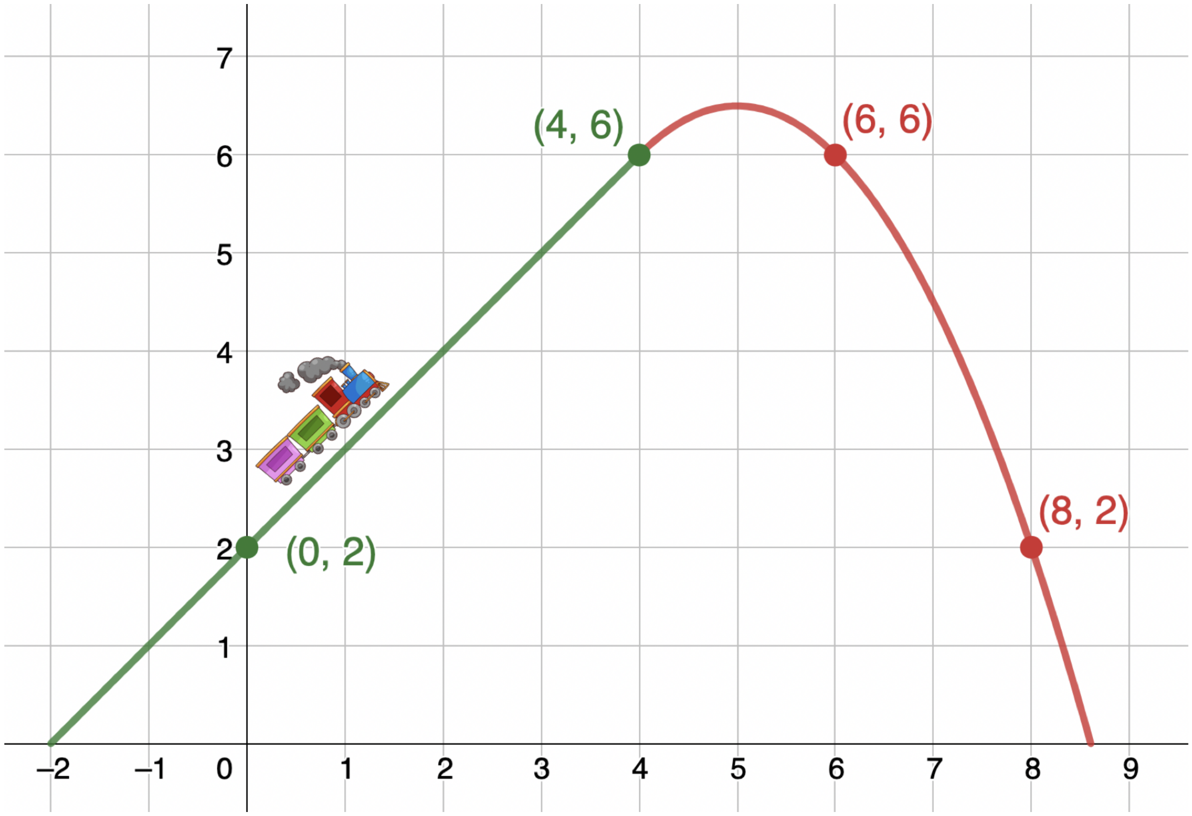
$$\begin{aligned}
\lim_{x\to 3} \frac{4g(x)}{1+h(x)} &= \frac{\lim_{x\to 3} 4g(x)}{\lim_{x\to 3}(1+h(x))} \\
&= \frac{4\lim_{x\to 3} g(x)}{\lim_{x\to 3} 1 + \lim_{x\to 3} h(x)} \\
&= \frac{4(4)}{1+1} \\
&= \frac{16}{2} \\
&= \boxed{8}
\end{aligned}$$

(c) Suppose $\lim_{x\to 2^-} s(x) = -3$ and $\lim_{x\to 2^+} s(x) = 3$. Use the algebraic limit laws to compute $\lim_{x\to 2} s(x)$ and $\lim_{x\to 2}(s(x)^2)$.

Since $\lim_{x\to 2^-} s(x) = -3$ and $\lim_{x\to 2^+} s(x) = 3$, $\lim_{x\to 2} s(x)$ $\boxed{\text{does not exist}}$. However, $\lim_{x\to 2^-} (s(x))^2 = (-3)^2 = 9$ and $\lim_{x\to 2^+} (s(x))^2 = (3)^2 = 9$, so $\lim_{x\to 2}(s(x))^2 = \boxed{9}$.

Problem 7. [12 points] As you may still remember from the First Midterm, the figure below shows a track of the Green-Red Mountain. Suppose the figure depicts the entire mountain, modeled by the function:

$$f(x) = \begin{cases} x+2, & x \leq 4 \\ -\frac{1}{2}x^2 + 5x - 6 & x > 4 \end{cases}$$



(a) Let $f(x)$ be a function. State the limit definition of the derivative function $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Compute the derivative of $x+2$ at $x=4$ and the derivative of $-\frac{1}{2}x^2+5x-6$ at $x=4$ using the limit definition. Verify that they are the same, and the track smoothly transitions from the green to the red. [Hint: compute the derivative function and plug in, as the derivative function will be used again in the next part.]

We can compute the derivative functions of $g(x)=x+2$ and $j(x)=-\frac{1}{2}x^2+5x-6$ via the limit in part (a):

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h+2)-(x+2)}{h} \\ &= \lim_{h\to 0}\frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} j'(x) &= \lim_{h\to 0}\frac{j(x+h)-j(x)}{h} \\ &= \lim_{h\to 0}\frac{-\frac{1}{2}(x+h)^2+5(x+h)-6-(-\frac{1}{2}x^2+5x-6)}{h} \\ &= \lim_{h\to 0}\frac{-\frac{1}{2}x^2-xh-\frac{1}{2}h^2+5x+5h-6+\frac{1}{2}x^2-5x+6}{h} \\ &= \lim_{h\to 0}\frac{-xh-\frac{1}{2}h^2+5h}{h} \\ &= \lim_{h\to 0}\frac{h(-x-\frac{1}{2}h+5)}{h} \\ &= \lim_{h\to 0}(-x-\frac{1}{2}h+5) \\ &= -x+5 \end{aligned}$$

Therefore, $\boxed{g'(4)=1 \text{ and } j'(4)=1}$

(c) Engineers are working to decide if it is safe for a train to travel on the mountain or if they need to build a tunnel through the mountain. Compute the slope of the tangent line to the mountain at the foot of the mountain, the point $(8,2)$, using the limit definition. If the absolute value of the slope is greater than 2, then it is too dangerous and the engineers need to build a tunnel. Do they need to build a tunnel? [Work must be shown to receive credit on this problem.]

By the work we did above to find $j'(x)$, we find that $j'(8)=-8+5=-3$, so the slope is too steep and the engineers need to build a tunnel.

☺ **Problem 8**. [8 points]

(a) For each of the following sequences, state whether they are arithmetic, geometric, neither, or both. Write down an **explicit** formula for each.

(i) $\{a_n\} = \{3, 7, 11, 15, 19, 23, 27, \ldots\}$

This sequence is $\boxed{\text{arithmetic}}$, with explicit formula $\boxed{a_n = 3 + 4(n-1)}$

(ii) $\{b_n\} = \{2, \frac{3}{7}, \frac{4}{49}, \frac{5}{343}, \frac{6}{2401}, \ldots\}$ (hint: $7 = 7^1$, $49 = 7^2$, $343 = 7^3$, $2401 = 7^4$)

This sequence is $\boxed{\text{neither}}$, with explicit formula $\boxed{a_n = \frac{n+1}{7^{n-1}}}$

(b) Suppose $c_n = \frac{3n^3+n-100}{2n^2-23n}$. Evaluate $\lim_{n\to\infty} c_n$.

Since the larger power of n is in the numerator, the limit is $\boxed{\infty}$

(c) Give an example of a sequence that is monotone but is not convergent.

The sequence a_n above is monotone, but not convergent.