MATH 1 FALL 2019 : LECTURE 02 WED 09-18-19

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Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! Today we are going to cover two important topics related to functions: 1) given two functions, how can we combine them to get a new function? and 2) even and odd functions, two very important classes of functions.

II. Combining Functions

Remarks

A fundamental idea that comes up in math over and over is combining simple building blocks to generate more complicated ones. For us, this means ways of combining functions to produce new functions.

Content

We will focus on five operations today for combining functions: $+, -, \times, /, \circ$. Suppose we have two functions, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The three easiest ways to combine functions are

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(f \times g)(x) = f(x) \times g(x)$

The next way to combine functions, $/$, is essentially the same, but we need to be worried about restricting the domain of the new function. We can define $(f/g)(x) = \dfrac{f(x)}{g(x)}$ in the same way as the above, but this new function will be undefined at the zeros of g.

The last way to combine functions is by composing them. As long as the codomain of one function matches the domain of the other, we can plug the output of one function as input into the other. The notation for function composition is different from what we expect; we read from the right to the left. So the new function $f \circ g$ is the one where we first do g, then apply f to the output.

Remarks

For which ways of combining functions does order not matter? Are the functions $(f+g)$ and $(g+f)$ equal? What about (f/g) and (g/f)? Or $f \circ g$ and $g \circ f$?

Remarks

A lot of these operations seem contrived, but are really natural.

- If I run a company, and want to figure out my net profit, I will just subtract my cost function from my revenue function.
- If I run several businesses, I can figure out my total revenue by summing the revenue functions of each business.
- If I run an ice cream truck, then my profit is some function of the number of people at the beach. But the number of people at the beach is a function of the temperature, so my profit is the composite of these two functions.

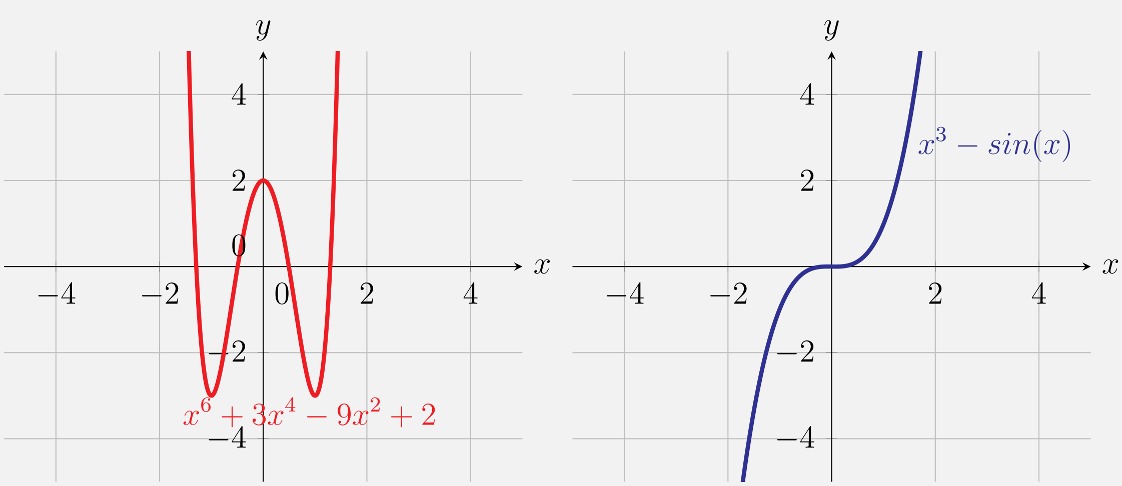
III. Properties of Functions

III.1. Even and Odd Functions.

Remarks

Symmetric functions are really nice! They look really nice, but even better, they have nice mathematical properties. We can tell a symmetric function by looking at the graph, but how can we tell algebraically?

Content



This first picture is visibly symmetric about the y-axis; this means if we reflected the picture across the y-axis, it would land back on itself. Similarly, the second picture is visibly symmetric about the origin, meaning if we rotate the picture 180° around the point $(0, 0)$ (also called the **origin**), then it would land back on itself.

Remarks

Can we have a function that is symmetric about the x-axis?

Content

Can we put the above symmetry statements into math terms?

- To be symmetric about the y-axis means that the value of the function at x and $-x$ is the same.
- To be symmetric about the origin means that the function at $-x$ is the negative of the function at x.

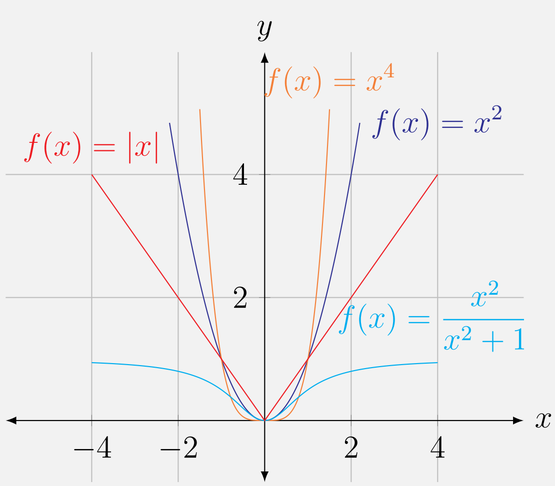
Definition

- If $f(x) = f(-x)$ for all x in the domain of f, then f is an **even function**. These are the functions that are symmetric about the y-axis.
- If $f(x) = -f(x)$ for all x in the domain of f, then f is an **odd function**. These are the functions that are symmetric about the origin.

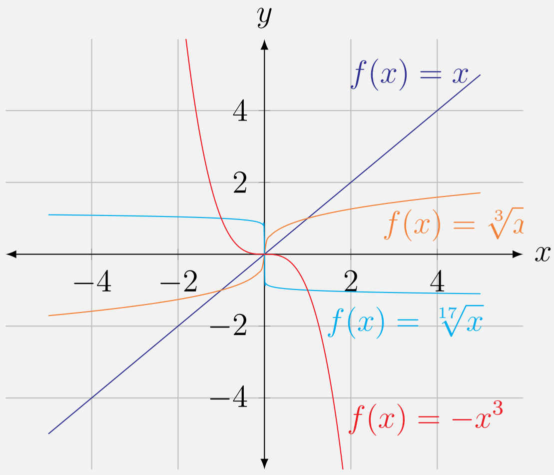
Remarks

Is every function even or odd? Why do you think they are called that?

Example



Example



Definition

The **absolute value function** $|\cdot| : \mathbb{R} \to \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Remarks

The absolute value function is very important in math; it measures distance. For example, $|x - 3| = 2$ means that x is a distance of 2 away from 3, so $x = 5$ or $x = 1$.

III.2. **Piecewise Functions.**

Definition

Some functions are defined by different algebraic formulations on different pieces of their domains, these functions are called **piecewise**.

Content

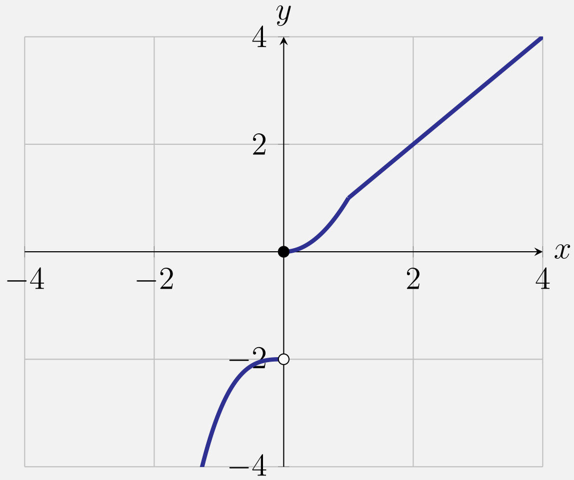
We can define functions in pieces; what is the example we have already seen of this? When graphing, we just graph piece by piece, paying close attention to what happens at the end of each piece. Generally, the graph uses open or closed circles to describe what happens at these end points.

Example

Consider the graph of

$$f(x) = \begin{cases} x^3 - 2 & x \in (-\infty, 0) \\ x^2 & x \in [0, 1] \\ x & x \in (1, \infty) \end{cases}$$

This is a good example of how piecewise functions can have different properties on different parts of their domain: this function is cubic in $(-\infty, 0)$, quadratic in $[0, 1]$, and linear in $(1, \infty)$. Piecewise functions can even be algebraic on some parts of their domain and transcendental on others.



Content

We need to take good care to make sure that piecewise functions are actually functions, and don't take on multiple output values on the ends of the intervals. For example,

$$f(x) = \begin{cases} x & x \in (-\infty, 1] \\ \dfrac{x^2}{2} & x \in [1, \infty) \end{cases}$$

is not a function because the input 1 maps to two different outputs.