MATH 1 FALL 2019 : LECTURE 06 FRI 09-27-19

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I. Big Idea Of The Day

Remarks

Functions describe the world! Lots of functions are periodic, meaning they repeat the same behavior over and over (seasonal movements, cyclical economic activity, predator and prey relationships, sleep cycle), and the simplest of these are the trigonometric functions. They are fundamental to everything we do in math.

II. Radians

Remarks

The trigonometric functions are some of the most important functions in the world! They come up very naturally in the measure of triangles and circles, which, as you know, are everywhere. They are very related to angles, so we need to learn a new way of measuring angles.

Remarks

Historically, you have probably measured angles in a unit called "degrees". There is a somewhat more natural measure of angles called "radian". Just like Celsius and Farenheit, these are different units to measure the same exact thing - angles.

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Consider a unit circle: a circle with radius 1. Given an angle θ, the arc it bounds has length s. The radian measure of θ is s; that is the angle that bounds an arc of length 1 has radian measure 1.

III. Six Basic Trigonometric Functions

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Just like we defined radians in terms of the unit circle, so will we define the trigonometric functions. Consider the unit circle, centered at the origin which we will call O, and another point, say $P = (x, y)$. We can consider the angle that has as it's initial side the positive x-axis from the origin to $(1, 0)$, and has as it's terminal side the line segment OP. Then we can define

$$\sin(\theta) = y$$
$$\cos(\theta) = x$$
$$\tan(\theta) = \frac{y}{x}$$

Remarks

The other three trigonometric functions are just the reciprocals of these:

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y}$$

We can also note by moving expressions around that $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$.

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If our triangle is bigger, say has hypotenuse of length r, we can inscribe it in the circle of radius r instead of the unit circle. We scale up the other two sides, then, to see that the point where it hits the circle is $P = (x, y) = (r\cos(\theta), r\sin(\theta))$, so $\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$. Some people remember this with the mnemonic

Sine is
Opposite over
Hypotenuse,
Cosine is
Adjacent over
Hypotenuse, and
Tangent is
Opposite over
Adjacent.

Remarks

There are some common values of sine and cosine most people just memorize, but we can get them by looking at common right triangles, or the Pythagorean theorem. Here is a table:

Angle	**Sine**	**Cosine**
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1	0

Example

Let's do an example of using these functions. I am cleaning out the gutters on my house, and have a 15 foot ladder. I only feel comfortable if it is placed at a 60° angle when I lean it against my house. How far away from my house should I place the base of the ladder?

IV. Graphs and Periods of the Trigonometric Functions

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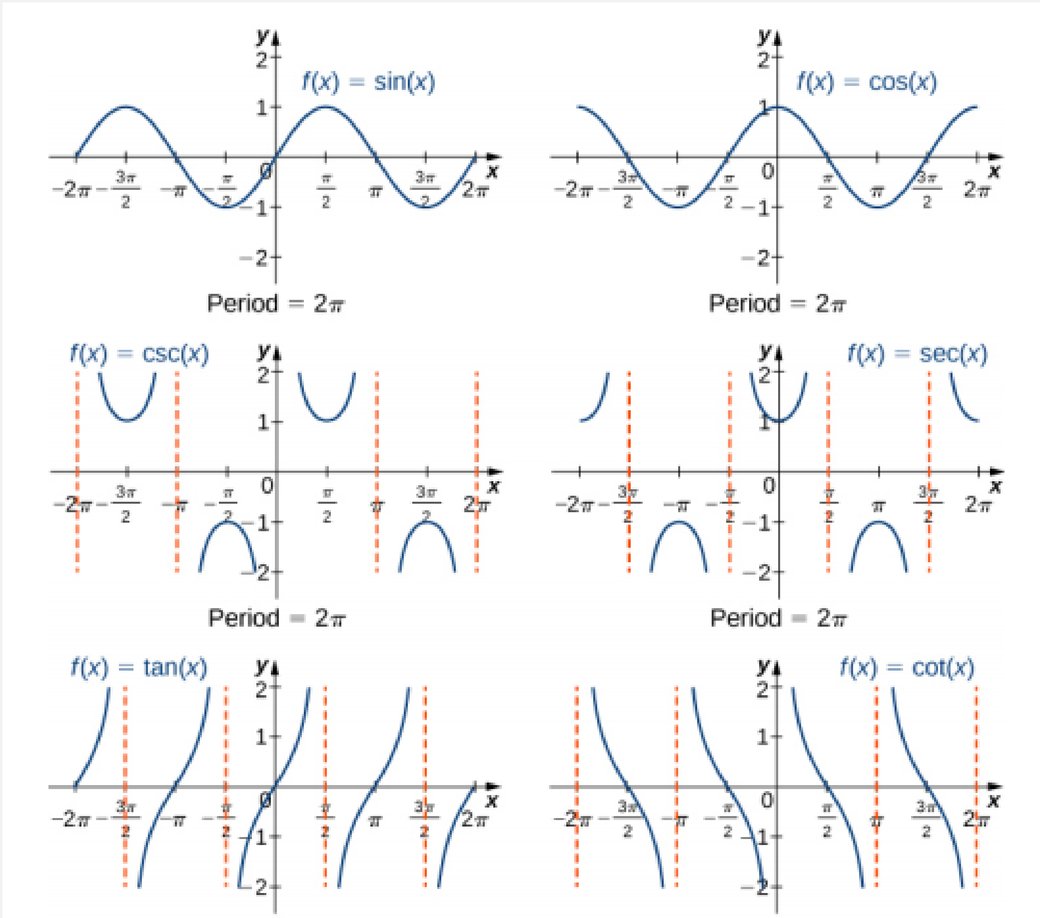
We have the definitions of $\sin(\theta) = y/r$, $\cos(\theta) = x/r$, $\tan(\theta) = y/x$, $\csc(\theta) = r/y$, $\sec(\theta) = x/r$, and $\cot(\theta) = x/y$. What are the domains of each of these functions? What happens to these functions if I plug in $\theta + 2\pi$? Or $\theta + 4\pi$? These functions are **periodic**, which means $f(x) = f(x + np)$ for some period p and all integers n. These functions have period 2π, because if I add any multiple of 2π (which corresponds to going around the circle some number of times), it doesn't change the output of the function.

Remarks

Using the fact that these functions are periodic, and the few values we have computed, we should be able to graph all of these. Let's do that now.



Content

Just like we learned last class, we can transform the trigonometric functions too! Let's consider $f(x) = A\sin(B(x - \alpha)) + C$. This does many things, let's treat them one at a time:

- Subtracting α from x shifts this graph to the right by α units.
- Scaling the input by B scales in the horizontal direction. This turns the period of our new function into $2\pi/|B|$.
- Scaling the output by A scales in the vertical direction, and we call $|A|$ the amplitude of $f(x)$.
- Finally, adding C to the output just moves $f(x)$ up by C units.

We can get pretty general periodic functions this way.

Example

Let's consider a city that reports they get the most sunlight with 15.7 hours on June 21, and the least sunlight with 8.3 hours on December 21. Then their daylight is modeled by $h(t) = 3.7\sin(\dfrac{2\pi}{365}(x - 80.5)) + 12$.

V. Trigonometric Identities

Remarks

We want to have ways to relate the trigonometric functions, so we can calculate more of them.

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Pythagorean Identities The Pythagorean theorem says that for a right triangle with sides x and y and hypotenuse r, we have $x^2 + y^2 = r^2$. If we divide both sides by r^2, we get $1 = \frac{y^2}{r^2} + \frac{x^2}{r^2} = (\frac{y}{r})^2 + (\frac{x}{r})^2 = \sin^2\theta + \cos^2\theta$. We could divide both sides by $\sin^2\theta$ to get $\csc^2\theta = 1 + \cot^2\theta$ or divide both sides by $\cos^2\theta$ to get $\sec^2\theta = 1 + \tan^2\theta$. These three equations are called the Pythagorean identites.

Addition Formulas If we know the values of trigonometric functions for two angles, we should be able to compute it for their sum. These formulas let us do that

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

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There are also simpler relations between sin and cos that are important to know, but aren't quite identities. One of these is if we plug in $-\theta$, we can see by reflecting our triangle that the x coordinate stays the same and the y coordinate is negated, so $\sin(-\theta) = -\sin(\theta)$, and $\cos(-\theta) = \cos(\theta)$. What happens when we shift an angle $\pi/2$? We can use the sum laws above to find that $\sin(\theta + \pi/2) = \cos(\theta)$, and $\cos(\theta + \pi/2) = -\sin(\theta)$. We can figure this out by shifting the graphs too!