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Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! Can we turn functions around? Functions map inputs to outputs - can we construct a new function that maps outputs to their corresponding inputs? These are called inverse functions, and the answer is maybe.

II. Inverse Functions

Definition

Given a function f with domain D and range R, its **inverse function** (if it exists) is a function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if $f(x) = y$.

Remarks

This means that if you compose a function and it's inverse, you get back the identity, so $f^{-1}(f(x)) = x$ for all $x \in D$, and $f(f^{-1}(y)) = y$ for all $y \in R$.
Note that we denote the inverse f^{-1}, but it is not an exponent and does not mean $1/f$!

Example

Let's fill out a table of values for $f(x) = x^2$, and what the inverse would need to do. Is $f^{-1}(x) = \pm\sqrt{x}$ a function? No, because each input value has two output values, so it fails the definition of a function.

Definition

We say a function is **one-to-one** if for $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$.

Content

One-to-one functions just mean two different inputs don't map to the same output under the function. They are special, because if a function is not one-to-one, it has no hope of having an inverse. We can check if functions are one-to-one if they pass the **horizontal line test**, which says that a function is one-to-one if no horizontal line intersects its graph more than once.

Content

To find an inverse function, we first need to determine if the function is one-to-one. If it is, then we set $f(x) = y$, and solve for x in terms of y. This gives us that $x = f^{-1}(y)$, and then because we usually call the input x and the output y, we switch the letters and write $y = f^{-1}(x)$.

III. Graphing Inverses

Content

If $f(a) = b$, then we know $f^{-1}(b) = a$. This means if (a, b) is a point on the graph of $f(x)$, then (b, a) is a point on the graph of $f^{-1}(x)$. So we just switch the x and y coordinates of all the points on the graph of $f(x)$. What does this mean for the graph of $f^{-1}(x)$? Switching x and y coordinates is the same as reflecting the graph across the line $y = x$, which explains why we want the horizontal line test.

IV. Restricted Domains and Inverse Trig Functions

Content

Tons of functions are not one-to-one, but are one-to-one on some portion of their domain. An example of this would be $f(x) = x^2$ which is not one-to-one, but which is one-to-one on either $[0, \infty)$ or or $(-\infty, 0]$. Essentially, we can define a new function by

$$g(x) = f(x) \text{ if } x \text{ is in some restricted portion of the domain of } f$$

Content

Are the trig functions one-to-one? They fail the horizontal line test. If we want to define inverses, then, we have to restrict the domain. Just so all mathematicians can agree, we all made the same choices:

- Inverse sine, denoted $\sin^{-1}$ or arcsin is defined as the inverse function to sine restricted to $[-\pi/2, \pi/2]$.
- Inverse cosine, denoted $\cos^{-1}$ or arccos is defined as the inverse function to cosine restricted to $[0, \pi]$.
- Inverse tangent, denoted $\tan^{-1}$ or arctan is defined as the inverse function to tangent restricted to $[-\pi/2, \pi/2]$.

- Inverse cotangent, denoted $\cot^{-1}$ or arccot is defined as the inverse function to cotangent restricted to $[0, \pi]$.
- Inverse cosecant, denoted $\csc^{-1}$ or arccsc is defined as the inverse function to cosecant restricted to $[-\pi/2, 0) \cup (0, \pi/2]$.
- Inverse secant, denoted $\sec^{-1}$ or arcsec is defined as the inverse function to secant restricted to $[0, \pi/2) \cup (\pi/2, \pi]$.

Remarks

Inverse functions don't always do what we want when we are on restricted domains, so we should be careful about that, with trig functions and others.

V. Exponential Functions

Remarks

Another incredibly important class of transcendental functions is exponential functions. A function $f(x)$ is an **exponential function** if $f(x) = b^x$ for some number $b > 0$, $b \neq 1$. We need $b > 0$ so we are always defined, and we need $b \neq 1$ or this function is boring. Where do these functions come up in the real world?

Content

What do we even mean by $f(x) = b^x$? It is clear if x is a positive integer, we can recall what it means if x is a negative integer, and we define $b^0 = 1$. If x is of the form $1/n$ for some integer n, it is just an n-th root. So we can say what we mean for x rational. For irrational, we just take limits, but it is too complex to go into right now. Let's practice graphing them.

Content

There are some rules of exponents we need to remember from algebra. For any $a > 0$, $b > 0$ and for all x and y:

- $b^x \cdot b^y = b^{x+y}$
- $\frac{b^x}{b^y} = b^{x-y}$
- $(b^x)^y = b^{xy}$
- $(ab)^x = a^x \cdot b^x$
- $(\frac{a}{b})^x = \frac{a^x}{b^x}$