MATH 1 FALL 2019 : LECTURE 08 WED 10-02-19

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I. Big Idea Of The Day

Remarks

Functions describe the world! We need a few more transcendental functions to fill out our toolkit, then we will really have lots of interesting functions to work with.

II. Exponential Functions

Remarks

Another incredibly important class of transcendental functions is exponential functions. A function $f(x)$ is an **exponential function** if $f(x) = b^x$ for some number $b > 0$, $b \neq 1$. We need $b > 0$ so we are always defined, and we need $b \neq 1$ or this function is boring. Where do these functions come up in the real world?

Content

What do we even mean by $f(x) = b^x$? It is clear if x is a positive integer, we can recall what it means if x is a negative integer, and we define $b^0 = 1$. If x is of the form $1/n$ for some integer n, it is just an n-th root. So we can say what we mean for x rational. For irrational, we just take limits, but it is too complex to go into right now. Let's practice graphing them.

Content

There are some rules of exponents we need to remember from algebra. For any $a > 0$, $b > 0$ and for all x and y:

- $b^x \cdot b^y = b^{x+y}$
- $\frac{b^x}{b^y} = b^{x-y}$
- $(b^x)^y = b^{xy}$
- $(ab)^x = a^x \cdot b^x$

- $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Remarks

The **natural exponential function**, $f(x) = e^x$, is a particularly important one. The slope of the tangent line to e^x is e^x. This is surprising and makes this function important!

III. Logarithmic Functions

Content

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$. So we can ask for its inverse function, which should have domain $(0, \infty)$ and range $(-\infty, \infty)$. We call this function the **logarithmic function with base** b, and denote it $\log_b$. Just as exponential functions were defined for any $b > 0$, $b \neq 1$, so is the logarithmic function. It is defined to be the inverse to the exponential function, so

$$\log_b x = y \text{ if and only if } b^y = x$$

Remarks

These functions are inverses! That means $\log_b b^x = x$ and $b^{\log_b x} = x$, because they undo each other.

Content

Just like the natural exponentional function is special, so too is the logarithm with base e, called the natural logarithm and denoted $\log_e$ or ln.

Content

Just like the rules of exponents, there are rules of logarithms that are similar, but harder to remember because logarithms are new to us. Suppose $a > 0$, $b > 0$, and $c > 0$, with $b \neq 1$, and r is any real number. Then:

$$\log_b(ac) = \log_b(a) + \log_b(c) \qquad \text{Product Property}$$

$$\log_b(\frac{a}{c}) = \log_b(a) - \log_b(c) \qquad \text{Quotient Property}$$

$$\log_b(a^r) = r \log_b(a) \qquad \text{Power Property}$$

Example

Let's do an example problem, and solve $\log_2(x^3) - \log_2(\sqrt{x}) = 4$. By the quotient property, we get that $4 = \log_2(\frac{x^3}{\sqrt{x}}) = \log_2(x^{5/2})$. By the power property, $4 = 5/2 \log_2(x)$, or $\log_2(x) = 8/5$. Thus $x = 2^{8/5}$.

Content

Lots of calculators only have log base 10 and natural log. This is because we can translate between different exponential functions and logarithmic functions simply.
For any $a > 0$, $b > 0$, and $a \neq 1$ and $b \neq 1$.
By the above log rules, we see that for any real number x

$$b^{x \log_b a} = b^{\log_b(a^x)} = a^x,$$

so we can change the base of an exponential function by including a log term in the exponent.
Similarly for any positive real number x,

$$\log_a x = \frac{\log_b x}{\log_b a}$$