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I. Big Idea Of The Day

Remarks

Functions describe the world! And to understand some behavior of functions we need to take a detour into a new topic: sequences. Sequences come up all the time in math and the real world, so they seem very natural to work with.

II. Terminology of Sequences

Definition

An **infinite sequence** is an ordered list of the form

$$a_1, a_2, a_3, \dots.$$

We sometimes write these as $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$. The subscript n is called the **index variable**, and each number a_n is a **term** of the sequence. Some sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function defined over the positive integers. In other cases, sequences are defined using a **recurrence relation**, where some of the terms of the sequence are given explicitly, and the rest of the terms are defined by the earlier terms.

Example

There are tons of examples of sequences. Can we come up with some?

- Suppose $a_n = n$. Then our sequence looks like $1, 2, 3, 4, 5, 6, \dots$. This is a sequence, it's not very interesting!
- The sequence $a_n = (-1)^n$. We need to be clear about our indexing here - let's say we start at $n = 1$. Then the sequence is $-1, 1, -1, 1, -1, \dots$.
- The sequence $a_1 = 2$, and $a_n = a_{n-1} + 5$. This is called an arithmetic sequence.
- The sequence $a_n = 5n + 2$ starting at $n = 0$. This is $2, 7, 12, 17, \dots$. We can go back and forth between recurrence relation and explicit formula.

- The sequence $a_n = 2 * (-1/3)^n$ starting at $n = 0$ is $2, -2/3, 2/9, -2/27, \ldots$. This is called a geometric sequence.
- The sequence $a_0 = 2$, $a_n = (-1/3) * a_{n-1}$ is $2, -2/3, 2/9, -2/27$. This is exactly the same as the last sequence! The same sequence can be written by explicit formula or by recurrence relation.
- The sequence $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$. These are the Fibonacci numbers, $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.
- The sequence $a_n = \dfrac{(-1)^n}{n}$ is $-1, 1/2, -1/3, 1/4, -1/5, \ldots$.

Definition

An **arithmetic sequence** is one where the difference between every pair of consecutive terms is the same. In general, these look like $a_n = cn + b$.
A **geometric sequence** is one where the quotient of any two consecutive terms is the same. In general, these look $a_n = cr^n$.

Content

A sequence can also be thought of as a function from the positive integers to $\mathbb{R}$. What are the inputs? What are the outputs? Now that we have a function, we can consider graphing the function, with inputs on the x-axis and outputs on the y-axis. Let's practice graphing some of our sequences from above.

Remarks

One thing we will want to do is turn sequences into explicit formulas. Mostly this is recognizing the pieces of them as arithmetic or geometric sequences, then putting those pieces together.

Example

Let's do a few examples:

- $-1/3, 2/9, -3/27, 4/81, -5/243, \ldots$ alternates in sign, so has a $(-1)^n$ in it. The numerator is an arithmetic sequence, $b_n = n$. The denominator is a geometric sequence, $c_n = 3^n$. Thus we have that this sequence is $a_n = \dfrac{(-1)^n n}{3^n}$.
- The sequence recursively defined by $a_1 = 1/2$, $a_n = a_{n-1} + (1/2)^n$ has first few terms $1/2, 3/4, 7/8, 15/16, \ldots$, so the denominator looks like $c_n = 2^n$, and the numerator is always just one lower, so $b_n = 2^n - 1$, So we have that $a_n = \dfrac{2^n - 1}{2^n} = 1 - \dfrac{1}{2^n}$.
- The sequence defined by $a_1 = -4$, $a_n = a_{n-1} + 6$. This sequence is $-4, 2, 8, 14, 20, \ldots$, an arithmetic sequence. It can be written $a_n = -4 + 6(n - 1) = -4 + 6n - 6 = -10 + 6n$.

III. Limits of Sequences

Remarks

It's hard to define limits without lots more math, so we will do our best here. Investigating what happens to a sequence as $n \to \infty$ is a fundamental question about sequences.

Example

Let's look at some of our example sequences and see what happens to them as $n \to \infty$.

- The sequence $a_n = 5n + 2$ starting at $n = 0$ is $2, 7, 12, 17, \ldots$. As $n \to \infty$, $a_n \to \infty$.
- The sequence $a_n = 1 - \dfrac{1}{2^n}$ is $1/2, 3/4, 7/8, 15/16, \ldots$ and we see as $n \to \infty$, $a_n \to 1$.
- The sequence $a_n = (-1)^n$ is $-1, 1, -1, 1, -1, \ldots$, and we see that a_n does not get closer and closer to anything as $n \to \infty$, it just alternates.
- The sequence $a_n = \dfrac{(-1)^n}{n}$ is $-1, 1/2, -1/3, 1/4, -1/5, \ldots$, and despite the fact the terms alternate, we see as $n \to \infty$, $a_n \to 0$.

Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. If this is the case, we write $\lim_{n \to \infty} a_n = L$.

A sequence which is not convergent is called a **divergent sequence**.

Definition

There is a more technical definition here too. A sequence $\{a_n\}$ converges to a finite number L if for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - L| < \epsilon$. We call L the limit of the sequence and write it just as we did above. If the sequence does not converge to a finite number, we call it divergent.

Remarks

What happens to a sequence doesn't depend on any finite number of terms! We can add a bunch of terms to the start of the sequence and it won't change whether the sequence has a limit, or what that limit is. Same if we remove a bunch of terms from the beginning of the sequence.

Remarks

Sequences can be divergent in different ways. Some of them are divergent because they go to infinity as n goes to infinity, or go to negative infinity as n goes to infinity. An example is $a_n = 3^n$, which diverges to infinity, and we write as $\lim_{n \to \infty} 3^n = \infty$, or

$a_n = 4 - 4n$ which diverges to negative infinity and which we write as $\lim_{n\to\infty} -4 - 4n = -\infty$.

Content

There are ways we can combine limits of functions we know to get limits of functions we don't know. Suppose we have two convergent sequences $\{a_n\}$ and $\{b_n\}$, so $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, and some real number c. Then we have the following algebraic limit laws:

- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$
- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
- $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A \cdot B$
- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{A}{B}$ provided $B \neq 0$ and each $b_n \neq 0$.

Example

We can find the limit of $a_n = 3 - 4/n^2$ using these laws, or $a_n = \dfrac{n^2+1}{3n^2 - 2n - 5}$.

Content

There is another way we can find limits of sequences, and it's by sandwiching a sequence between two others we already know the limit of. It is called the Squeeze Theorem, and says if we have two convergent sequences $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty c_n}$, and a third sequence that is between $\{a_n\}$ and $\{c_n\}$ for sufficiently large n, i.e. there is some $N \in \mathbb{N}$ such that for $n \geq N$, we have $a_n \leq b_n \leq c_n$, then $\lim_{n\to\infty} b_n = L$.

This is a ton of math, but just means we are squishing one sequence between two others that converge to L, so the given sequence must converge to L.