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Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! And to understand some behavior of functions we need to take a detour into a new topic: sequences. Sequences come up all the time in math and the real world, so they seem very natural to work with.

II. Limits of Sequences

Remarks

It's hard to define limits without lots more math, so we will do our best here. Investigating what happens to a sequence as $n \to \infty$ is a fundamental question about sequences.

Example

Let's look at some of our example sequences and see what happens to them as $n \to \infty$.

- The sequence $a_n = 5n + 2$ starting at $n = 0$ is $2, 7, 12, 17, \ldots$. As $n \to \infty$, $a_n \to \infty$.
- The sequence $a_n = 1 - \dfrac{1}{2^n}$ is $1/2, 3/4, 7/8, 15/16, \ldots$ and we see as $n \to \infty$, $a_n \to 1$.
- The sequence $a_n = (-1)^n$ is $-1, 1, -1, 1, -1, \ldots$, and we see that a_n does not get closer and closer to anything as $n \to \infty$, it just alternates.
- The sequence $a_n = \dfrac{(-1)^n}{n}$ is $-1, 1/2, -1/3, 1/4, -1/5, \ldots$, and despite the fact the terms alternate, we see as $n \to \infty$, $a_n \to 0$.

Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. If this is the case, we write $\lim_{n\to\infty} a_n = L$.

A sequence which is not convergent is called a **divergent sequence**.

Definition

There is a more technical definition here too. A sequence $\{a_n\}$ converges to a finite number L if for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - L| < \epsilon$. We call L the limit of the sequence and write it just as we did above. If the sequence does not converge to a finite number, we call it divergent.

Remarks

What happens to a sequence doesn't depend on any finite number of terms! We can add a bunch of terms to the start of the sequence and it won't change whether the sequence has a limit, or what that limit is. Same if we remove a bunch of terms from the beginning of the sequence.

Remarks

Sequences can be divergent in different ways. Some of them are divergent because they go to infinity as n goes to infinity, or go to negative infinity as n goes to infinity. An example is $a_n = 3^n$, which diverges to infinity, and we write as $\lim_{n\to\infty} 3^n = \infty$, or $a_n = 4 - 4n$ which diverges to negative infinity and which we write as $\lim_{n\to\infty} -4 - 4n = -\infty$.

Content

There are ways we can combine limits of functions we know to get limits of functions we don't know. Suppose we have two convergent sequences $\{a_n\}$ and $\{b_n\}$, so $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, and some real number c. Then we have the following algebraic limit laws:

- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$
- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
- $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A \cdot B$
- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{A}{B}$ provided $B \neq 0$ and each $b_n \neq 0$.

Example

We can find the limit of $a_n = 3 - 4/n^2$ using these laws, or $a_n = \dfrac{n^2+1}{3n^2 - 2n - 5}$.

Content

There is another way we can find limits of sequences, and it's by sandwiching a sequence between two others we already know the limit of. It is called the Squeeze Theorem, and says if we have two convergent sequences $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty c_n}$, and a third sequence that is between $\{a_n\}$ and $\{c_n\}$ for sufficiently large n, i.e. there is some $N \in \mathbb{N}$ such that for $n \geq N$, we have $a_n \leq b_n \leq c_n$, then $\lim_{n\to\infty} b_n = L$.

This is a ton of math, but just means we are squishing one sequence between two others that converge to L, so the given sequence must converge to L.