MATH 1 FALL 2019 : LECTURE 11 WED 10-09-19

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Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! And to understand some behavior of functions we need to take a detour into a new topic: sequences. Sequences come up all the time in math and the real world, so they seem very natural to work with.

II. Limits of Sequences

Remarks

It's hard to define limits without lots more math, so we will do our best here. Investigating what happens to a sequence as $n \to \infty$ is a fundamental question about sequences.

Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. If this is the case, we write $\lim\limits_{n\to\infty} a_n = L$.

A sequence which is not convergent is called a **divergent sequence**.

Remarks

What happens to a sequence doesn't depend on any finite number of terms! We can add a bunch of terms to the start of the sequence and it won't change whether the sequence has a limit, or what that limit is. Same if we remove a bunch of terms from the beginning of the sequence.

Remarks

Sequences can be divergent in different ways. Some of them are divergent because they go to infinity as n goes to infinity, or go to negative infinity as n goes to infinity. An example is $a_n = 3^n$, which diverges to infinity, and we write as $\lim_{n\to\infty} 3^n = \infty$, or $a_n = 4 - 4n$ which diverges to negative infinity and which we write as $\lim_{n\to\infty} -4 - 4n = -\infty$.

Content

There is another way we can find limits of sequences, and it's by sandwiching a sequence between two others we already know the limit of. It is called the Squeeze Theorem, and says if we have two convergent sequences $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty c_n}$, and a third sequence that is between $\{a_n\}$ and $\{c_n\}$ for sufficiently large n, i.e. there is some $N \in \mathbb{N}$ such that for $n \geq N$, we have $a_n \leq b_n \leq c_n$, then $\lim_{n\to\infty} b_n = L$.

This is a ton of math, but just means we are squishing one sequence between two others that converge to L, so the given sequence must converge to L.

III. Bounded and Monotonic Sequences

Definition

- A sequence $\{a_n\}$ is **bounded below** if every term in it is greater than or equal to some real number M, that is $a_n \geq M$ for some real number M.
- A sequence $\{a_n\}$ is **bounded above** if every term in it is less than or equal to some real number M, that is $a_n \leq M$ for some real number M.
- A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.
- A sequence $\{a_n\}$ which is not bounded below or bounded above is called **unbounded.**

Remarks

Boundedness is some statement about what is happening out at infinity, as n gets large. If there are only big terms at the start of the sequence, then the function is bounded. Thus if a sequence is unbounded, there are larger and larger magnitude terms as n gets large, which means the sequence can't converge! Thus we have the following theorem.

Content

Theorem III.0.1. *If a sequence is convergent, it is bounded.*

Definition

- A sequence is increasing for $n \geq n_0$ if $a_n \leq a_{n+1}$ for all $n \geq n_0$.
- A sequence is decreasing for $n \geq n_0$ if $a_n \geq a_{n+1}$ for all $n \geq n_0$.
- A sequence is a monotone sequence for $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

Content

Theorem III.0.2 (Monotone Convergence Theorem)**.** *If $\{a_n\}$ is a bounded sequence, and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then a_n converges.*

We can't prove this, but we can use it, and draw a picture to understand it.

Example

We can use the monotone convergence theorem to prove something we have taken for granted, that $\lim_{n\to\infty} \frac{1}{2^n} = 0$.