MATH 1 FALL 2019 : LECTURE 13 MON 10-14-19

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Contents

I. Big Idea Of The Day 1
II. Evaluating Limits 1
III. Continuity 3

I. Big Idea Of The Day

Remarks

Functions describe the world! Functions can do wacky things, so we need to define a notion of the limit for a function and compute some.

II. Evaluating Limits

Content

Let $f(x)$ be a function defined on an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that L is the limit of $f(x)$ as x approaches a. We can write this as $\lim_{x\to a} f(x) = a$.

Remarks

Realistically, we calculate limits to start by writing down a table of values and calculating, then seeing if the values approach some fixed number. We will show ways to find more complicated limits without approximation later.

Content

There are some limits we will just memorize!

- If n is positive and even, then $\lim_{x\to a} \dfrac{1}{(x-a)^n} = \infty$.
- If n is positive and odd, then $\lim_{x\to a^-} \dfrac{1}{(x-a)^n} = -\infty$ and $\lim_{x\to a^+} \dfrac{1}{(x-a)^n} = \infty$.
- $\lim_{x\to a} x = a$
- $\lim_{x\to a} c = c$ for some $f(x) = c$ a real number.

Remarks

We can evaluate limits graphically, by looking at the graph of a function, but we also can do it algebraically. We need a few simple limits and rules for combining them.

Content

Two simple limits: $\lim_{x\to a} x = a$, and $\lim_{x\to a} c = c$ where c is some constant. Then there are limit laws. Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ in some open interval containing a, with $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Suppose c is some real number. Then

- $\lim_{\mathrm{x}\to\mathrm{a}} f(x) + g(x) = \lim_{\mathrm{x}\to\mathrm{a}} f(x) + \lim_{\mathrm{x}\to\mathrm{a}} g(x) = L + M$
- $\lim_{\mathrm{x}\to\mathrm{a}} f(x) - g(x) = \lim_{\mathrm{x}\to\mathrm{a}} f(x) - \lim_{\mathrm{x}\to\mathrm{a}} g(x) = L - M$
- $\lim_{\mathrm{x}\to\mathrm{a}} f(x) \cdot g(x) = \lim_{\mathrm{x}\to\mathrm{a}} f(x) \cdot \lim_{\mathrm{x}\to\mathrm{a}} g(x)$
- $\lim_{\mathrm{x}\to\mathrm{a}} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{\mathrm{x}\to\mathrm{a}} f(x)}{\lim_{\mathrm{x}\to\mathrm{a}} g(x)} = \dfrac{L}{M}$ as long as $M \neq 0$
- $\lim_{\mathrm{x}\to\mathrm{a}} cf(x) = c \lim_{\mathrm{x}\to\mathrm{a}} f(x) = cL$
- $\lim_{\mathrm{x}\to\mathrm{a}} (f(x)^n) = (\lim_{\mathrm{x}\to\mathrm{a}} f(x))^n = L^n$ for every positive integer n
- $\lim_{\mathrm{x}\to\mathrm{a}} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{\mathrm{x}\to\mathrm{a}} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

Remarks

This tells us exactly what we should do for polynomials - just plug in the value. We get the following theorem: Let $p(x)$ and $q(x)$ be polynomials, and a a real number. Then $\lim_{x\to a} p(x) = p(a)$, and $\lim_{x\to a} p(x)/q(x) = p(a)/q(a)$ so long as $q(a) \neq 0$.

Content

Since the value of the limit does not depend on the value of the function at the point, we can often manipulate the function to try to come up with one that is defined and equal everywhere except the point, then use this to calculate the limit. Problem solving strategy for when the limit has the indeterminate form $0/0$:

- Make sure our function has the appropriate form and cannot be evaluated immediately using the limit laws.
- Find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$. Try
  - If $f(x)$ and $g(x)$ are polynomials, factor and cancel.
  - If $f(x)$ or $g(x)$ contains a square root containing x, try multiplying by the conjugate.
  - If $f(x)/g(x)$ is a complex fraction, simplify it.
- Apply the limit laws.

Example

- $\lim_{x\to 3} \dfrac{x^2 - 3x}{2x^2 - 5x - 3} = 3/7$

- $\lim_{x\to 4} \dfrac{2-\sqrt{x}}{4-x} = 1/4$
- $\lim_{x\to 1} \dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = -1/4$

Content

We can also want to evaluate limits of the form $K/0$ for $K \neq 0$. Separate the part that becomes infinite in magnitude and then apply the limit laws.

Example

- $\lim_{x\to 2^-} \dfrac{x-3}{x^2-2x} = \infty$
- $\lim_{x\to 0} 1/x + 5/(x(x-5)) = -1/5$

Remarks

We don't know how to find many limits yet. What is $\lim_{x\to 0} \cos x$? We can't get it from our algebraic limit laws.

III. Continuity

Remarks

Just like limits depend on the point they are being evaluated at, continuity is a property of a function at a point! We are headed towards continuous functions, which are functions that are continuous at every point, but never forget that continuity is just a property at a point.

Remarks

Continuous functions are, intuitively, functions we can draw without taking our pencil off the paper.

Content

There are some things that can prohibit this. Let's look at what it means for a function to be continuous at a point a. First of all, we need a to exist! Second of all, if I draw from the left, it should meet up with my drawing from the right, so I need the limit of $f(x)$ as x approaches a to exist. Finally, since we know the limit of $f(x)$ as x approaches a doesn't depend on $f(a)$ at all, we need to assert that $\lim_{x\to a} f(x) = f(a)$. This is the definition of continuity at a.