MATH 1 FALL 2019 : LECTURE 13 MON 10-14-19

SAMUEL TRIPP

Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! Evaluating limits is really hard, except for polynomials where we can just plug in the value. What other functions can we just evaluate limits by plugging in the value? Continuous functions are one broad class of such functions.

II. Continuity

Remarks

Just like limits depend on the point they are being evaluated at, continuity is a property of a function at a point! We are headed towards continuous functions, which are functions that are continuous at every point, but never forget that continuity is just a property at a point.

Remarks

Continuous functions are, intuitively, functions we can draw without taking our pencil off the paper.

Content

There are some things that can prohibit this. Let's look at what it means for a function to be continuous at a point a. First of all, we need a to exist! Second of all, if I draw from the left, it should meet up with my drawing from the right, so I need the limit of $f(x)$ as x approaches a to exist. Finally, since we know the limit of $f(x)$ as x approaches a doesn't depend on $f(a)$ at all, we need to assert that $\lim_{x \to a} f(x) = f(a)$. This is the definition of continuity at a. A function is discontinuous at a if it is not continuous at a.

Content

How do we check if a function $f(x)$ is continuous at a point a?

- Is $f(a)$ defined? If no, the function is discontinuous at a.
- Does $\lim_{x\to a} f(x)$ exist? If no, the function is discontinuous at a.
- Does $\lim_{x\to a} f(x) = f(a)$? If no, the function is discontinuous at a.
- If yes to all of the above, the function is continuous at a.

Remarks

We already showed that a ton of functions are continuous on their domains - polynomials and rational functions. We showed last class that for polynomials $p(x)$ and $q(x)$, $\lim_{x\to a} p(x) = p(a)$ and $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, which is exactly the definition.

III. Types of Discontinuities

Remarks

Often times we define a nice property in math, then immediately consider what can happen to things that don't satisfy that property. We have seen this with limits (diverging to infinity, for example), and we do it here for the different types of ways things can be discontinuous.

Content

If $f(x)$ is discontinuous at a, then

- f has a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists.
- f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but are not equal.
- f has a **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

Example

$(x^2-4)/(x-2)$, piecewise function, or $(x+2)/(x+1)$

IV. Continuity on Intervals

Remarks

Continuity is a property of a function at a point. We want to extend this thinking to define what it means for a function to be continuous on an interval. Intuitively, it means we can get from any point in the interval to any other point in the interval without picking up our pencil. First, we need two definitions.

Definition

- A function $f(x)$ is **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.
- A function $f(x)$ is **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$.

Content

- A function is continuous on an open interval (a, b) if it is continuous at every point in the interval.
- A function is continuous on a closed interval $[a, b]$ if it is continuous at every point in the interval (a, b) and is continuous from the left at b and continuous from the right at a.
- A function is continuous on a mixed interval $(a, b]$ if it is continuous at every point in the interval (a, b) and is continuous from the left at b.
- A function is continuous on a mixed interval $[a, b)$ if it is continuous at every point in the interval (a, b) and is continuous from the right at a.

V. Squeeze Theorem

Remarks

We need one more tool to compute limits and establish that some functions are continuous, namely the trig functions. It is the Squeeze Theorem! But applied to functions.

Content

Theorem V.0.1. *Suppose $f(x)$, $g(x)$, and $h(x)$ are functions defined for all $x \neq a$ on an open interval containing a. If $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ on an open interval containing a, and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$, where L is a real number, then $\lim_{x \to a} g(x) = L$.*