MATH 1 FALL 2019 : LECTURE 13 MON 10-14-19

SAMUEL TRIPP

Contents

I. Big Idea Of The Day 1
II. Continuity 1
III. Composite Function Theorem 2
IV. Intermediate Value Theorem 3
V. Defining the Derivative 3

I. Big Idea Of The Day

Remarks

Functions describe the world! Continuous functions are great. Today we will learn two more reasons why: the Composite Function Theorem, and the Intermediate Value Theorem.

II. Continuity

Content

Let's recall the definition of a function being continuous at a point. We say $f(x)$ is continuous at the point a if $f(a)$ exists, $\lim_{x\to a} f(x)$ exists, and these values are equal. We can also define continuity on an interval by saying that the function has to be continuous at every point in the open interval, and continuous from the correct directions at the interval's endpoints, if they are included.

Remarks

We had a brief aside to show that $\lim_{x\to 0}\sin(x) = 0$, so $\sin(x)$ is continuous at 0. By the algebraic limit laws, we know $\cos(x) = \sqrt{1-\sin^2(x)}$, so $\lim_{x\to 0}\cos(x) = \lim_{x\to 0}\sqrt{1-\sin^2(x)} = \sqrt{\lim_{x\to 0} 1-\sin^2(x)} = \sqrt{\lim_{x\to 0} 1 - \lim_{x\to 0}\sin^2(x)} = \sqrt{1-(\lim_{x\to 0}\sin(x))^2} = \sqrt{1-0} = 1$, so $\cos(x)$ is continuous at 0 also!

Remarks

We can also show an important limit using Squeeze Theorem. Even if you don't follow the derivation, you should memorize the limit. We can compute $\lim_{x \to 0} \frac{\sin x}{x} = 1$. It relies on an augmented triangle and the fact that these are even functions.

Example

Let's figure out the intervals on which $f(x) = \frac{x-1}{x^2+2x}$ is continuous. Remember that we said polynomials and rational functions are continuous at every point in their domains, so $f(x)$ is continuous at all real numbers except the roots of $x^2 + 2x$. By factoring into $x(2+x)$, we see that this means $f(x)$ is continuous on $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$.

III. Composite Function Theorem

Remarks

The most useful tool in computing limits is this theorem, which says that I can swap the order of limits and continuous functions.

Content

Theorem III.0.1. *If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(L)$.*

Example

What is $\lim_{x \to \pi/2} \cos(x - \pi/2)$? Well $\lim_{x \to \pi/2} x - \pi/2 = 0$, and cos is continuous at 0, so by the composite function theorem, $\lim_{x \to \pi/2} \cos(x - \pi/2) = \cos(\lim_{x \to \pi/2} x - \pi/2) = \cos(0) = 1$.

Content

With this, we can prove that trigonometric functions are continuous on their entire domains. We will show this for $\cos(x)$, note that it is exactly the same for $\sin(x)$, and recall that the other four trig functions are just ratios of these, so by the algebraic limit laws, are continuous on their domains. We want to show $\lim_{x \to a} \cos(x) = \cos(a)$. Someone along the way came up with the really clever idea of writing $x = (x - a) + a$.

Then we can see what to do:

$$\begin{aligned}
\lim_{x\to a}\cos(x) &= \lim_{x\to a}\cos((x-a)+a)\\
&= \lim_{x\to a}\big[\cos(x-a)\cos(a)-\sin(x-a)\sin(a)\big]\\
&= \lim_{x\to a}\cos(x-a)\cos(a)-\lim_{x\to a}\sin(x-a)\sin(a)\\
&= \lim_{x\to a}\cos(x-a)\lim_{x\to a}\cos(a)-\lim_{x\to a}\sin(x-a)\lim_{x\to a}\sin(a)\\
&= \cos(\lim_{x\to a}(x-a))\cos(a)-\sin(\lim_{x\to a}(x-a))\sin(a)\\
&= \cos(a)
\end{aligned}$$

This shows that $\cos(x)$ is continuous on its whole domain. We could do the same angle sum trick with $\sin(x)$ to show that it is continuous on its whole domain, although we will not.

IV. Intermediate Value Theorem

Remarks

Functions that are continuous on a closed interval of the form $[a,b]$ for $a,b\in\mathbb{R}$ are really nice! One of the useful properties they satisfy is the intermediate value theorem.

Content

Theorem IV.0.1 (Intermediate Value Theorem). *Let $f(x)$ be continuous on the closed interval $[a,b]$ for some $a,b\in\mathbb{R}$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ satisfying $f(c)=z$.*

This theorem is a lot of math for something that is straightforward in pictures! If I have to connect $(a,f(a))$ to $(b,f(b))$ on a graph without picking up my pencil, I will have to cross a point with y-coordinate z for any z between $f(a)$ and $f(b)$. That's all this theorem says.

Example

One common use of the Intermediate Value Theorem is to check if equations have solutions. If you can find some value where they are negative, say a, and some value where they are positive, say b, and the function $f(x)$ is continuous on $[a,b]$, then there is some $c\in[a,b]$ where $f(c)=0$.
Is there some number where $x=\cos(x)$? This is the same as asking if $f(x)=x-\cos(x)$ has any roots. Well $f(0)=-1$ and $f(\pi/2)=\pi/2$, and $f(x)$ is continuous on $[0,\pi/2]$, so there are solutions.

Remarks

The Intermediate Value Theorem says nothing about values z outside of the range from $f(a)$ to $f(b)$! And we can only apply it when $f(x)$ is continuous on $[a,b]$.

V. Defining the Derivative

Remarks

To think about the derivative, we have to go back to our discussion of secant lines from before!

Content

We can use the slope of secant lines to $f(x)$ at $(a, f(a))$ to estimate the rate of change of $f(x)$ near a. We do this by choosing another point, $(x, f(x))$, and calculating the slope of the secant line that connects these to points, or

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

This is a better and better estimate if x is really close to a, which sometimes we write as $x = a + h$ for really small h, then we get

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}.$$

Definition

We have a name for these things; if $f(x)$ is defined on an interval I containing a, then for $x \neq a$ in I, $Q = \dfrac{f(x) - f(a)}{x - a}$ is called a **difference quotient**. Equivalently, if $h \neq 0$ is chosen so that $a + h$ is in I, then $Q = \dfrac{f(a+h) - f(a)}{h}$ is called a **difference quotient** with increment h.

Content

These difference quotients become better and better approximations of the slope of the tangent line as $x \to a$, or $h \to 0$. But we know now how to deal with "getting closer and closer to": limits! So we make the following definition.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The **tangent line** to $f(x)$ at a is the line passing through $(a, f(a))$ with slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.

Equivalently, we may define the **tangent line** to $f(x)$ at a as the line passing through $(a, f(a))$ with slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Definition

This is the derivative. Let $f(x)$ be a function defined on an open interval containing a. Then the **derivative** of $f(x)$ at a, which we write $f'(a)$, is the slope of the tangent line to $f(x)$ at a, if it exists. Equivalently, $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, provided this limit exists.