MATH 1 FALL 2019 : LECTURE 13 MON 10-14-19

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Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! Now that we have limits and continuous functions, we turn our attention to the central topic of this course: derivatives! We need to define what they are, discuss their physical interpretation, and figure out how to compute them.

II. Defining the Derivative

Remarks

To think about the derivative, we have to go back to our discussion of secant lines from before!

Content

We can use the slope of secant lines to $f(x)$ at $(a, f(a))$ to estimate the rate of change of $f(x)$ near a. We do this by choosing another point, $(x, f(x))$, and calculating the slope of the secant line that connects these to points, or

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

This is a better and better estimate if x is really close to a, which sometimes we write as $x = a + h$ for really small h, then we get

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}.$$

Definition

We have a name for these things; if $f(x)$ is defined on an interval I containing a, then for $x \neq a$ in I, $Q = \dfrac{f(x) - f(a)}{x - a}$ is called a **difference quotient**. Equivalently, if $h \neq 0$ is chosen so that $a + h$ is in I, then $Q = \dfrac{f(a+h) - f(a)}{h}$ is called a **difference quotient** with increment h.

Content

These difference quotients become better and better approximations of the slope of the tangent line as $x \to a$, or $h \to 0$. But we know now how to deal with "getting closer and closer to": limits! So we make the following definition.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The **tangent line** to $f(x)$ at a is the line passing through $(a, f(a))$ with slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.
Equivalently, we may define the **tangent line** to $f(x)$ at a as the line passing through $(a, f(a))$ with slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Definition

This is the derivative. Let $f(x)$ be a function defined on an open interval containing a. Then the **derivative** of $f(x)$ at a, which we write $f'(a)$, is the slope of the tangent line to $f(x)$ at a, if it exists. Equivalently, $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, provided this limit exists.

III. Examples

Example

Using both definitions of the derivative, let's compute $f'(2)$ for $f(x) = x^2$.
We see $f'(2) = \lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2} \dfrac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} x + 2 = 4$.
Similarly, $f'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h} = \lim_{h \to 0} \dfrac{4 + 4h + h^2 - 6}{h} = \lim_{h \to 0} \dfrac{h(4+h)}{h} = \lim_{h \to 0} 4 + h = 4$.

Example

Compute the tangent line to $g(x) = 1/x$ at $x = 3$.
We know that the slope of the tangent line to $g(x)$ at $x = 3$ is just $g'(x)$, so we get $m_{\text{tan}} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} = \lim_{x \to 3} 3 - x3x(x-3) = \lim_{x \to 3} \frac{-1}{3x} = -1/9$. And we know a point on the line, so we are done.

IV. Rates of Change and Velocities

Content

The **derivative** of $f(x)$ at a, the **slope of the tangent line** to $f(x)$ at a and the **instantaneous rate of change** of $f(x)$ at a are three different ways of saying the same exact thing.

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One of the most common uses for the derivative is to apply it to a function which describes the position of an object, to capture the instantaneous velocity of the object. The rate of change of position is exactly the same as the velocity, so the derivative, which is the instantaneous rate of change, captures the instantaneous velocity.

Example

Suppose a ball is dropped from a cliff with a height of 64 feet, and it's height is modeled by $s(t) = -16t^2 + 64$. What is the velocity of the rock 1 second after it is dropped?
We know by the above that velocity at a specific time is the same as the instantaneous rate of change, or the derivative. So we get that $f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{-16x^2 + 64 - 48}{x - 1} = \lim_{x \to 1} \frac{-16x^2 + 16}{x - 1} = \lim_{x \to 1} \frac{(-16)(x-1)(x+1)}{x - 1} = \lim_{x \to 1} -16(x + 1) = -32\text{ft/s}$.

V. The Derivative as a Function

Remarks

Functions are just a set of inputs, a set of outputs, and a rule for assigning exactly one output to each input. The derivative function for $f(x)$ takes as input real numbers, and gives as output $f'(a)$, wherever it is defined.

Definition

Let f be a function. Then the **derivative function**, f', is the function whose domain consists of the values of x for which the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Definition

- A function f is said to be **differentiable at** a if $f'(a)$ exists.
- A function f is said to be **differentiable on** S if it is differentiable at every point in an open set S.
- A function f is said to be **differentiable** if it is differentiable at every point in it's domain.

Example

Find the derivative function of $f(x) = x^2 + 3x + 2$. We know $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h + 2 - x^2 - 3x - 2}{h} = \lim_{h \to 0} \frac{h(2x + h + 3)}{h} = \lim_{h \to 0} 2x + h + 3 = 2x + 3$.

Example

Find the derivative function of $g(x) = \sqrt{x}$. We know $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(h)}{h} = \lim_{h \to 0} \frac{\sqrt[2]{x+h} - \sqrt[2]{x}}{h} = \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

Content

There is a ton of notation we use to all mean the derivative. If we write $f(x)$ for the function, we can denote the derivative function by $\frac{d}{dx}(f(x))$ or $f'(x)$. If we write y for the function, we can denote the derivative function by y' or $\frac{dy}{dx}$. This is all just notation that means the same exact thing.