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Contents

I. Big Idea Of The Day

Remarks

Functions describe the world! Now that we have limits and continuous functions, we turn our attention to the central topic of this course: derivatives! We need to define what they are, discuss their physical interpretation, and figure out how to compute them.

II. Defining the Derivative

Definition

Let $f(x)$ be a function defined in an open interval containing a. The **tangent line** to $f(x)$ at a is the line passing through $(a, f(a))$ with slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.
Equivalently, we may define the **tangent line** to $f(x)$ at a as the line passing through $(a, f(a))$ with slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Definition

This is the derivative. Let $f(x)$ be a function defined on an open interval containing a. Then the **derivative** of $f(x)$ at a, which we write $f'(a)$, is the slope of the tangent line to $f(x)$ at a, if it exists. Equivalently, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, provided this limit exists.

III. The Derivative as a Function

Remarks

Functions are just a set of inputs, a set of outputs, and a rule for assigning exactly one output to each input. The derivative function for $f(x)$ takes as input real numbers, and gives as output $f'(a)$, wherever it is defined.

Definition

Let f be a function. Then the **derivative function**, f', is the function whose domain consists of the values of x for which the following limit exists:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Definition

- A function f is said to be **differentiable at** a if $f'(a)$ exists.
- A function f is said to be **differentiable on** S if it is differentiable at every point in an open set S.
- A function f is said to be **differentiable** if it is differentiable at every point in it's domain.

Example

Find the derivative function of $f(x) = x^2 + 3x + 2$. We know $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h + 2 - x^2 - 3x - 2}{h} = \lim_{h\to 0} \frac{h(2x+h+3)}{h} = \lim_{h\to 0} 2x + h + 3 = 2x + 3.$

Example

Find the derivative function of $g(x) = \sqrt{x}$. We know $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(h)}{h} = \lim_{h\to 0} \frac{\sqrt[2]{x+h} - \sqrt[2]{x}}{h} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

Content

There is a ton of notation we use to all mean the derivative. If we write $f(x)$ for the function, we can denote the derivative function by $\frac{d}{dx}(f(x))$ or $f'(x)$. If we write y for the function, we can denote the derivative function by y' or $\frac{dy}{dx}$. This is all just notation that means the same exact thing.

IV. Graph of the Derivative Function

Remarks

The derivative of a function tells us the instantaneous rate of change of the function; thus we can figure out it's sign without even knowing the original function. Either by direct computation, or by approximation, we can graph the derivative of a function.

Example

Let's graph a few of the functions and derivatives we have computed, $f(x) = x^2+3x+2$ and $g(x) = \sqrt{x}$, and see what the relationship between the function and its derivative tells us.

Example

Let's look at another example: $r(x) = 10x(x-1)(x+1)$. Can we graph the derivative function, r'? Can we compute the derivative? Let's: $\lim_{h\to 0} \frac{r(x+h)-r(x)}{h} = \lim_{h\to 0} \frac{(10h^3+30h^2x+30hx^2-10h+10x^3-10x)-(10x^3-10x)}{h} = \lim_{h\to 0} \frac{h(10h^2+30hx+30x^2-10)}{h} = \lim_{h\to 0} 10h^2+30hx+30x^2-10 = 30x^2-10$.