MATH 1 FALL 2019 : LECTURE 18 FRI 10-25-19

SAMUEL TRIPP

Contents

I. Big Idea Of The Day

Remarks

No big idea today! Just reviewing all the big ideas of limits, continuity, and derivatives.

II. Differentiable Functions are Continuous

Remarks

We want to prove one more thing before we review. Our intuitive definition is that if a function is differentiable at a point, it can be drawn without sharp corners. This should imply it is continuous. Let's prove that.

Content

Suppose $f(x)$ is differentiable at a. Then $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists. We want to show $f(x)$ is continuous a, or $\lim_{x\to a} f(x) = f(a)$. By adding 0 and multiplying by 1 we get

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x) - f(a) + f(a)) \\
&= \lim_{x\to a}(\frac{f(x)-f(a)}{x-a} \cdot (x-a) + f(a)) \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) + \lim_{x\to a} f(a) \\
&= f'(a) \cdot 0 + f(a) = f(a)
\end{aligned}$$

III. Limits

Remarks

Continuity and differentiability are both about limits! Limits of functions are very important. Let's remember some and practice.

Content

There are some limits we just memorize! For some real numbers a and c, and some positive integer n:

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} \frac{1}{(x-a)^n} = \infty \text{ if } n \text{ is even}$$

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty \text{ and } \lim_{x \to a^+} \frac{1}{(x-a)^n} = \infty \text{ if } n \text{ is odd}$$

Example

- What is $\displaystyle\lim_{y \to 5} \frac{1}{y-5}$?
- What is $\displaystyle\lim_{t \to -1^+} \frac{1}{t+1} + 4$?

Content

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then limits (including one-sided limits) obey the following laws:

(1) Sum/Difference Law: $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

(2) Product Law: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

(3) Quotient Law: $\displaystyle\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$, (as long as $M \neq 0$)

(4) Power Law: $\lim_{x \to a} f(x)^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$, (if n is a positive integer)

(5) Root Law: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$, (for all L if n is odd and for $L \geq 0$ if n is even)

Example

Suppose $\lim_{x \to 3} f(x) = 4$ and $f(3) = 10$. Evaluate $\displaystyle\lim_{x \to 3} \frac{\sqrt{17 - f(x)^2}}{f(x) + 2}$.

Content

This is enough to show us that for polynomials and rational functions, the limits are found just by plugging in, if that is defined. We see for $p(x)$ and $q(x)$ polynomials, $\lim_{x\to a} p(x) = p(a)$ and $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$.

Content

If L or M are $\pm\infty$, then we can still apply the above rules, subject to the following rules about ∞. If $c > 0$ is a positive real number:

- $\infty \pm c = \infty$
- $\infty \pm 0 = \infty$
- $\infty \cdot c = \infty$
- $\infty \cdot -1 = -\infty$
- $\frac{1}{\infty} = 0$
- $\infty + \infty = \infty$
- $\infty \cdot \infty = \infty$

Remarks

We have to be worried about indeterminate forms! If we try to apply the algebraic limit laws and get any of the following forms, we cannot evaluate the limit in that way and need to try something clever.

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0 \cdot \infty,\ 1^{\infty},\ \infty - \infty,\ 0^0,\ \infty^0$$

Example

Evaluate $\lim_{x\to 1}((x-1) \cdot \frac{1}{(x-1)})$.

Content

If we get $\frac{0}{0}$ when we try to evaluate a limit, the book suggests three techniques:

i. factor and cancel
ii. multiply the numerator and denominator by the conjugate of an expression involving a square root
iii. simplify complex fractions

Example

- Evaluate $\lim_{x\to 1} \frac{x^2-1}{x^2+x-2}$.
- Evaluate $\lim_{h\to 0} \frac{\sqrt{16+h}-\sqrt{16}}{h}$.

- Evaluate $\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Content

If we get $\frac{K}{0}$ for some $K \neq 0$ for our limit by plugging in, try to factor out the part that leads to zero on the denominator. We can evaluate this limit using our $\frac{1}{(x-a)^n}$ laws, and we can plug in for the rest.

Example

Evaluate $\lim_{x\to 4^-} \frac{x^2+x-2}{x^2-3x-4}$.

Content

The final method we have for finding limits is the Squeeze Theorem, which says if we have some function that is stuck between two other functions in a neighborhood of a point, and the outer functions have the same limit value at the point, then so does the original function. If $f(x) \leq g(x) \leq h(x)$ in an open interval containing a, and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, THEN $\lim_{x\to a} g(x) = L$.

Example

Consider the function

$$f(x) = \begin{cases} 1+2x^2 \text{ if } x \text{ is rational} \\ 1+x^4 \text{ if } x \text{ is irrational} \end{cases}$$

Prove that $\lim_{x\to 0} f(x) = 1$.

IV. Continuity

Remarks

Limits are nice, but don't at all depend on the value of the function at the point. Some functions have the really nice property at a point that the limit of the function at that point is equal to the value of the function at that point. These functions are called continuous.

Content

A function $f(x)$ is **continuous** at a if

i. $f(a)$ is defined,

ii. $\lim_{x\to a} f(x)$ exists, and

iii. $\lim_{x \to a} f(x) = f(a)$.

Content

Polynomials are continuous everywhere. Rational functions are only discontinuous where the denominator equals zero. Trig functions are continuous everywhere they are defined.

Content

There are three types of discontinuities we focus on. If $f(x)$ is discontinuous at a, then:

- f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. This is essentially just a hole in a graph.
- f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, but they are not equal. This looks like the graph jumps.
- f has a **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$. This looks like the function heads off to positive or negative infinity.

Example

We can consider each of these types of discontinuity. What type of discontinuity is:

- $f(\theta) = \csc\theta$ at $\theta = \pi$?
- $f(x) = \dfrac{x^2 - 2x}{x}$ at $x = 0$?
- $f(x) = \dfrac{|x|}{x}$ at $x = 0$?

Content

If $f(x)$ and $g(x)$ are continuous on an interval, then so are $f + g$, $f - g$, $f \cdot g$, f/g (except where $g(x) = 0$), $f \circ g$, and $g \circ f$. Continuous functions can be combined to get new continuous functions.

Content

There are two really important functions that we learned.

Theorem IV.0.1 (Composite Function Theorem)**.** *Suppose $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$. Then $\lim_{x \to a} f(x) = f(\lim_{x \to a} g(x)) = f(L)$.*

Theorem IV.0.2 (Intermediate Value Theorem)**.** *Suppose $f(x)$ is continuous on the interval $[a, b]$, and z is in between $f(a)$ and $f(b)$. Then there is some c in the interval $[a, b]$ such that $f(c) = z$.*

Example

Show that for $f(x) = \cos(x) + \sin(x) + x^2$, $f(x) = 5$ has a solution.

V. Derivatives

Content

The derivative of the function at a point is defined to be the slope of the tangent line at that point if it exists, and is the same as asking for the instantaneous rate of change of the function at that point. We see that the derivative of $f(x)$ at a, which we denote $f'(a)$ is defined to be the slope of the tangent line, which is the limit of the slopes of the secant lines through a and a nearby point as the point gets closer and closer. Thus we have

$$\begin{aligned} f'(a) &= \mathrm{m}_{\tan} \text{ at } a \\ &= \lim_{x\to a} \mathrm{m}_{\sec} \text{ through } (a, f(a)) \text{ and } (x, f(x)) \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \end{aligned}$$

If instead of calling the nearby point x, we call it $a + h$, we get that $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$.

Remarks

Continuity is really just a process in finding limits. From the above, we see that derivatives are also just a process in finding limits. Limits are really important!

Example

- Let $f(x) = x^2 - 2x$ and find $f'(5)$.
- Real world example. Sam's Math Store can sell calculators at a price of $p(x) = -0.01x + 200$ per calculator. It costs $c(x) = 50x + 5000$ to produce calculators. What is Sam's profit when he sells 5,000 calculators? What is the rate of change of profit? Should he increase or decrease his production?

Content

Instead of finding the derivative at every point individually, we can find the derivative of the function as a new function itself. Given a function $f(x)$, the derivative function is defined to be $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, whose domain is the values where that limit exists.

Example

Before we found for $f(x) = x^2 - 2x$, $f'(5)$. Let's instead find $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h\to 0} 2x+h-2 = 2x-2$. Now we can compute $f'(5) = 8$, which matches what we got before, which is good. But now we can compute $f'(2)$ and $f'(-3/4)$ and f' of anything much faster than doing it one by one!

Content

The derivative function has the value at every point equal to the slope of the tangent line to the original function at that point. That is to say that $f'(a)$ is the slope of the tangent line to $f(x)$ at a. So if the slope of $f(x)$ is negative at a, $f'(a) < 0$, and if the slope of $f(x)$ is positive at a, $f'(a) > 0$, and if the slope at $f(x)$ is zero at a, then $f'(a) = 0$.

Content

A function $f(x)$ is called differentiable at a if $f'(a)$ exists. A function $f(x)$ is called differentiable if $f'(a)$ exists for every a in the domain of $f(x)$.