MATH 1 FALL 2019 : LECTURE 19 MON 10-28-19

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Contents

I. Big Idea Of The Day

Remarks

Derivatives are just limits, and limits are hard to compute. Can we figure out the derivatives of a bunch of common functions, and how to combine them, so that we can compute a bunch of limits of harder functions without doing limits.

II. Constant Rule

Content

Consider the function $f(x) = c$. What is the rate of change of this function at any point? 0! So we should have $f'(x) = 0$. Let's check. We can compute $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$, as desired. This is the **constant rule**: if c is some constant real number and $f(x) = c$, then $f'(x) = 0$.

III. Power Rule

Content

We have found that for $f(x) = x^2$, $f'(x) = 2x$. Let's compute $g'(x)$ for $g(x) = x^3$. We get $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = 3x^2$. This suggests what we hope the derivative of a power function.

Content

Let $f(x) = x^n$. Let's first compute $(x+h)^n = x^n + nx^{n-1}h + h^2(\ldots\text{other terms}\ldots)$. Then $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{x^n + nx^{n-1}h + h^2(\ldots\text{other terms}\ldots) - x^n}{h} = \lim_{h\to 0} nx^{n-1} + h(\ldots\text{other terms}\ldots) = nx^{n-1}$. So we have derived the **power rule**: if n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

IV. Sum and Difference Rules

Remarks

There are many different notations for the derivative. To avoid writing the derivative of a sum function as $(f+g)'(x)$, we will write $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)+g(x))$. But these are just different notation for the same thing: $f+g$ is some function, and $(f+g)'(x)$ and $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)+g(x))$ are both the derivative function to this function.

Content

Let's compute $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)+g(x))$. By plugging in to the definition and simplifying and applying limit laws, we get that $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)+g(x)) = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) + \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) = f'(x)+g'(x)$. This is the **sum rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)+g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}f(x) + \frac{\mathrm{d}}{\mathrm{dx}}g(x)$. This is how we would expect sums to interact with derivatives, and is exactly how differences work too. The **difference rule** is that $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)-g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}f(x) - \frac{\mathrm{d}}{\mathrm{dx}}g(x)$.

Example

For $f(x) = x^5 + x^2 - 3$, compute $f'(x)$.

Content

Let's do one more derivative rule then figure out the derivative of any polynomial. We see for some constant c, $\frac{\mathrm{d}}{\mathrm{dx}} cf(x) = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h\to 0} c\frac{f(x+h) - f(x)}{h} = c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = c\frac{\mathrm{d}}{\mathrm{dx}} f(x)$, which is the **constant multiple rule**.

Example

Now we can compute the derivative of any polynomial: if $f(x) = 4x^3 - 2x^2 + 19x - 3$, what is $f'(x)$? If $g(x) = \pi x^4 + x^3 - \frac{3}{5}x^2 - 9$, what is $g'(x)$?

V. Differentiable Functions are Continuous

Remarks

While we are in the mood of proving things, let's prove that differentiable functions are continuous.

Content

Suppose $f(x)$ is differentiable at a. Then $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ exists. We want to show $f(x)$ is continuous a, or $\lim_{x\to a} f(x) = f(a)$. By adding 0 and multiplying by 1 we get

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x) - f(a) + f(a)) \\ &= \lim_{x\to a}(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a)) \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) + \lim_{x\to a} f(a) \\ &= f'(a) \cdot 0 + f(a) = f(a)\end{aligned}$$

VI. Product Rule

Remarks

This is the first rule that we have learned this whole class that doesn't work the way we want! Let's look at $\frac{\mathrm{d}}{\mathrm{dx}}(x^2) = 2x$. We want to have that $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)g(x)) = \frac{\mathrm{d}}{\mathrm{dx}} f(x) \frac{\mathrm{d}}{\mathrm{dx}} g(x)$, but this would mean $2x = \frac{\mathrm{d}}{\mathrm{dx}}(x^2) = \frac{\mathrm{d}}{\mathrm{dx}}(x) \cdot \frac{\mathrm{d}}{\mathrm{dx}}(x) = 1 \cdot 1 = 1$, which it doesn't. So our product rule must be more complicated.

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Let's just compute, in a clever way by adding zero. We get $\frac{\mathrm{d}}{\mathrm{dx}}(f(x) \cdot g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x))}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \frac{g(x+h) - g(x)}{h} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \lim_{h\to 0} f(x) = f'(x)g(x) + g'(x)f(x) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \cdot g(x) + \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) \cdot f(x)$. This is the **product rule**: $\frac{\mathrm{d}}{\mathrm{dx}} f(x)g(x) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \cdot g(x) + \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) \cdot f(x)$.

VII. Quotient Rule

Content

Again we see that $\frac{\mathrm{d}}{\mathrm{dx}} f(x)/g(x)$ is not equal to $\frac{\mathrm{d}}{\mathrm{dx}} f(x) / \frac{\mathrm{d}}{\mathrm{dx}} g(x)$, by considering $\frac{\mathrm{d}}{\mathrm{dx}} x^3/x = \frac{\mathrm{d}}{\mathrm{dx}} x^2 = 2x$, but $\frac{\mathrm{d}}{\mathrm{dx}} x^3 / \frac{\mathrm{d}}{\mathrm{dx}} x = 3x^2/1 = 3x^2$. We need a more complicated rule! We get the **quotient rule**: $\frac{\mathrm{d}}{\mathrm{dx}} \frac{f(x)}{g(x)} = \frac{\frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \cdot g(x) - f(x) \cdot \frac{\mathrm{d}}{\mathrm{dx}}(g(x))}{(g(x))^2}$.

Content

With this, we can extend the power rule to the negative integers too. The **extended power rule**: if n is any integer, $\frac{\mathrm{d}}{\mathrm{dx}} x^n = nx^{n-1}$.