MATH 1 FALL 2019 : LECTURE 19 MON 10-28-19

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I. Big Idea Of The Day

Remarks

Derivatives are just limits, and limits are hard to compute. Can we figure out the derivatives of a bunch of common functions, and how to combine them, so that we can compute a bunch of limits of harder functions without doing limits. We might even look at derivatives of trig functions today.

II. Rules!

Content

- **Constant rule**: if c is some constant real number, and $f(x) = c$, then $f'(x) = 0$.
- **Power rule**: if n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- **Sum and Difference rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) = f'(x) \pm g'(x)$
- **Product rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)g(x)) = f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x)) + \frac{\mathrm{d}}{\mathrm{dx}}(f(x))g(x) = f(x)g'(x) + f'(x)g(x)$
- **Quotient rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{dx}}(f(x)) - f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x))}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Content

Motivation for the quotient rule is the derivative of $f(x) = x^2 = x^3/x$. We see $2x \neq 3x^2$, so we can't just divide derivatives. We will not prove the quotient rule.

III. Practice Examples

Example

- Let $f(x) = (x^2 + 3)(2x^2 - x - 9)$. Compute $f'(x)$.
- Let $g(x) = (x - 2)(x^4 - 3x + 5)$. Compute $g'(x)$.
- Let $h(x) = (x^5 + x^4 + x^3 + x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)$. Compute $h'(x)$.
- Let $p(x) = \dfrac{2x}{4x^2 + 3}$. Compute $p'(x)$.
- Let $q(x) = \dfrac{7x^3 - 2x}{x - 9}$. Compute $q'(x)$.
- Let $r(x) = \dfrac{(x - 3)}{(x + 3)(x + 4)}$. Compute $r'(x)$.

IV. Extended Power Rule

Content

We only know how to differentiate x^n if n is a positive integer. Let's figure it out for $x^{-n} = \dfrac{1}{x^n}$. We can just apply the quotient rule to get $\dfrac{\mathrm{d}}{\mathrm{dx}}(x^{-n}) = \dfrac{\mathrm{d}}{\mathrm{dx}}(\dfrac{1}{x^n}) = \dfrac{x^n \cdot 0 - nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$, so our power rule is exactly the same for any integer: $\dfrac{\mathrm{d}}{\mathrm{dx}}(x^n) = nx^{n-1}$ regardless of if n is positive or negative.

Example

Let $f(x) = \dfrac{6}{x^4}$. Compute $f'(x)$.

V. A Couple Trigonometric Limits and Derivatives of Trig Functions

Remarks

So far, we can only compute derivatives of polynomials and rational functions (although we are pretty good at that now!). We want to be able to do more: trig functions, exponentials, and logs. We will head towards being able to compute derivatives of trig functions today.

Content

To compute derivatives of trig functions, we need a few important limits. The first is $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. We will just take this one as believable, but we can graph it to get intuition for why it is true. From this, we can derive the other important limit: $\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$. Let's compute:

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(x)}{x} &= \lim_{x\to 0}\frac{1-\cos(x)}{x}\cdot\frac{1+\cos(x)}{1+\cos(x)} \\
&= \lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0}\frac{\sin^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0}\frac{\sin(x)}{x}\lim_{x\to 0}\frac{sin(x)}{1+\cos(x)} = 1\cdot\frac{0}{2} = 0
\end{aligned}$$

Content

Now let's compute $\frac{\mathrm{d}}{\mathrm{dx}}(\sin(x))$. We get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}(\sin(x)) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0}(\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\sin(h)\cos(x)}{h}) \\
&= \lim_{h\to 0}(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos(x)
\end{aligned}$$

We could do a very similar thing to compute that $\frac{\mathrm{d}}{\mathrm{dx}}(\cos(x)) = -\sin(x)$. Do these make sense graphically?

Example

Now we can compute the derivative of $f(x) = 4x^2\sin(x)$ and $g(x) = \frac{x^3+3x-7}{\cos(x)}$ easily. Furthermore, since $h(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, we can compute $h'(x)$ using the quotient rule.

Content

In this theme, we can compute all of the derivatives of trig functions using the quotient rule, but we will just write them down here.

- $\frac{\mathrm{d}}{\mathrm{dx}}(\tan(x)) = \sec^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\cot(x)) = -\csc^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\sec(x)) = \sec(x)\tan(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\csc(x)) = -\csc(x)\cot(x)$