MATH 1 FALL 2019 : LECTURE 21 FRI 11-01-19

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Contents

I. Big Idea Of The Day

Remarks

Derivatives are just limits, and limits are hard to compute. Today we will figure out limits of trig functions, then see where to go next.

II. Rules!

Content

- **Constant rule**: if c is some constant real number, and $f(x) = c$, then $f'(x) = 0$.
- **Power rule**: if n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- **Sum and Difference rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) = f'(x) \pm g'(x)$
- **Product rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)g(x)) = f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x)) + \frac{\mathrm{d}}{\mathrm{dx}}(f(x))g(x) = f(x)g'(x) + f'(x)g(x)$
- **Quotient rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{dx}}(f(x)) - f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x))}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

III. A Couple Trigonometric Limits and Derivatives of Trig Functions

Remarks

So far, we can only compute derivatives of polynomials and rational functions (although we are pretty good at that now!). We want to be able to do more: trig functions, exponentials, and logs. We will head towards being able to compute derivatives of trig functions today.

Content

To compute derivatives of trig functions, we need a few important limits. The first is $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. We will just take this one as believable, but we can graph it to get intuition for why it is true. From this, we can derive the other important limit: $\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$. Let's compute:

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(x)}{x} &= \lim_{x\to 0}\frac{1-\cos(x)}{x}\cdot\frac{1+\cos(x)}{1+\cos(x)} \\
&= \lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0}\frac{\sin^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0}\frac{\sin(x)}{x}\lim_{x\to 0}\frac{sin(x)}{1+\cos(x)} = 1\cdot\frac{0}{2} = 0
\end{aligned}$$

Content

Now let's compute $\frac{\mathrm{d}}{\mathrm{dx}}(\sin(x))$. We get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}(\sin(x)) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0}(\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\sin(h)\cos(x)}{h}) \\
&= \lim_{h\to 0}(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos(x)
\end{aligned}$$

We could do a very similar thing to compute that $\frac{\mathrm{d}}{\mathrm{dx}}(\cos(x)) = -\sin(x)$. Do these make sense graphically?

Example

Now we can compute the derivative of $f(x) = 4x^2\sin(x)$ and $g(x) = \dfrac{x^3+3x-7}{\cos(x)}$ easily. Furthermore, since $h(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$, we can compute $h'(x)$ using the quotient rule.

Content

In this theme, we can compute all of the derivatives of trig functions using the quotient rule, but we will just write them down here.

- $\dfrac{\mathrm{d}}{\mathrm{dx}}(\tan(x)) = \sec^2(x)$
- $\dfrac{\mathrm{d}}{\mathrm{dx}}(\cot(x)) = -\csc^2(x)$
- $\dfrac{\mathrm{d}}{\mathrm{dx}}(\sec(x)) = \sec(x)\tan(x)$
- $\dfrac{\mathrm{d}}{\mathrm{dx}}(\csc(x)) = -\csc(x)\cot(x)$

IV. Chain Rule

Remarks

We can figure out derivatives of polynomials and trigonometric functions, and any sums, differences, products, or quotients, but we don't yet know how to compute derivatives of compositions of functions. So we need a chain rule.

Remarks

Consider the function $f(x) = \cos(x^3)$. We want to know $f'(x)$, which is the rate of change of f with respect to x. As we change a little in the x-direction, it will cause a change in x^3, and then cos will change; it is a chain reaction. But the above says the derivative should depend on the rate of change of x^3 with respect to x, and the derivative of $\cos(x^3)$ with respect to x^3.

Content

Let's compute. We get $f'(a) = \lim\limits_{x\to a} \dfrac{\cos(x^3)-\cos(a^3)}{x-a}$, which we can multiply by $\dfrac{x^3-a^3}{x^3-a^3}$. This gets us that $f'(a) = \lim\limits_{x\to a} \dfrac{\cos(x^3)-\cos(a^3)}{x^3-a^3} \lim\limits_{x\to a} \dfrac{x^3-a^3}{x-a} = \lim\limits_{x^3\to a^3} \dfrac{\cos(x^3)-\cos(a^3)}{x^3-a^3} \lim\limits_{x\to a} \dfrac{x^3-a^3}{x-a}$. The second term is $3a^2$. The first term is $\lim\limits_{u\to a^3} \dfrac{\cos(u)-\cos(a^3)}{u-a^3} = \dfrac{\mathrm{d}}{\mathrm{dx}}(\cos(u))$ evaluated at a^3, so we have our whole derivative is $f'(a) = \cos(a^3)\cdot 3a^2$.

Content

This is the chain rule. We will not prove it, but it is very important. Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function $h(x) = (f \circ g)(x) = f(g(x))$ is given by $h'(x) = f'(g(x))g'(x)$.
Another way to say this is that if y is a function of u, and u is a function of x, then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}}\frac{\mathrm{du}}{\mathrm{dx}}$.

Content

Recall that this means if we want to compute the derivative of a composition, we need to clearly identify what we are composing before we do the derivative. When doing these problems, it is worth doing out the work clearly! Write down what $f(x)$ is, write down what $g(x)$ is, and make sure your function of note is the composition in the correct order!

Example

Now we can actually compute $f'(x)$ for $f(x) = \cos(x^3)$. It should match what we got above.
Another example would be $g(x) = \frac{1}{(3x^2+1)^2}$.
What about $h(x) = \sin^3(x) = (\sin(x))^3$?
Let's do more than two functions: $f(x) = (\cos(x^2+4x+2))^7 + 4$.
Let's do an odd one, $g(x) = \sec(x^5+3x^3)$.
How about Leibniz notation for $y = \tan(4x^2-3x+1)$.