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I. Big Idea Of The Day

Remarks

We can compute derivatives of polynomial and rational functions without using the limit definition, and trig functions, and sums, differences, products, and quotients of these. Let's figure out a rule to compute derivatives of compositions without using the limit definition.

II. Summary of What We Can Differentiate

Content

- **Constant rule**: if c is some constant real number, and $f(x) = c$, then $f'(x) = 0$.
- **Power rule**: if n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- **Sum and Difference rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) = f'(x) \pm g'(x)$
- **Product rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)g(x)) = f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x)) + \frac{\mathrm{d}}{\mathrm{dx}}(f(x))g(x) = f(x)g'(x) + f'(x)g(x)$
- **Quotient rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{dx}}(f(x)) - f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x))}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
- **Chain rule**: To come later.

We also learned that $\frac{\mathrm{d}}{\mathrm{dx}}\sin(x) = \cos(x)$ and $\frac{\mathrm{d}}{\mathrm{dx}}\cos(x) = -\sin(x)$, so we can compute the derivatives of all trigonometric functions using the quotient rule (or can memorize them!). For posterity:

- $\frac{\mathrm{d}}{\mathrm{dx}}(\tan(x)) = \sec^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\cot(x)) = -\csc^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\sec(x)) = \sec(x)\tan(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\csc(x)) = -\csc(x)\cot(x)$

III. Chain Rule

Remarks

We can figure out derivatives of polynomials and trigonometric functions, and any sums, differences, products, or quotients, but we don't yet know how to compute derivatives of compositions of functions. So we need a chain rule.

Remarks

Consider the function $f(x) = \cos(x^3)$. We want to know $f'(x)$, which is the rate of change of f with respect to x. As we change a little in the x-direction, it will cause a change in x^3, and then cos will change; it is a chain reaction. But the above says the derivative should depend on the rate of change of x^3 with respect to x, and the derivative of $\cos(x^3)$ with respect to x^3.

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Let's compute. We get $f'(a) = \lim_{x\to a} \frac{\cos(x^3) - \cos(a^3)}{x - a}$, which we can multiply by $\frac{x^3 - a^3}{x^3 - a^3}$. This gets us that $f'(a) = \lim_{x\to a} \frac{\cos(x^3) - \cos(a^3)}{x^3 - a^3} \lim_{x\to a} \frac{x^3 - a^3}{x - a} = \lim_{x^3\to a^3} \frac{\cos(x^3) - \cos(a^3)}{x^3 - a^3} \lim_{x\to a} \frac{x^3 - a^3}{x - a}$. The second term is $3a^2$. The first term is $\lim_{u\to a^3} \frac{\cos(u) - \cos(a^3)}{u - a^3} = \frac{\mathrm{d}}{\mathrm{dx}}(\cos(u))$ evaluated at a^3, so we have our whole derivative is $f'(a) = \cos(a^3) \cdot 3a^2$.

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This is the chain rule. We will not prove it, but it is very important. Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function $h(x) = (f \circ g)(x) = f(g(x))$ is given by $h'(x) = f'(g(x))g'(x)$.
Another way to say this is that if y is a function of u, and u is a function of x, then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}}\frac{\mathrm{du}}{\mathrm{dx}}$.

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Recall that this means if we want to compute the derivative of a composition, we need to clearly identify what we are composing before we do the derivative. When doing these problems, it is worth doing out the work clearly! Write down what $f(x)$ is, write down what $g(x)$ is, and make sure your function of note is the composition in the correct order!

Example

Now we can actually compute $f'(x)$ for $f(x) = \cos(x^3)$. It should match what we got above.
Another example would be $g(x) = \dfrac{1}{(3x^2+1)^2}$.
What about $h(x) = \sin^3(x) = (\sin(x))^3$?
Let's do more than two functions: $f(x) = (\cos(x^2+4x+2))^7 + 4$.
Let's do an odd one, $g(x) = \sec(x^5+3x^3)$.
How about Leibniz notation for $y = \tan(4x^2-3x+1)$.

IV. Derivatives of Inverse Functions

Remarks

Derivatives are slopes of tangent lines. If we take a function $f(x)$, then the slope of the tangent line at a $f'(a)$ should be very related to the slope of the tangent line of the inverse function, because the inverse function is just a reflection over the line $y = x$. As a matter of fact, the slope of the tangent line to the inverse function is the reciprocal of the slope of the tangent line to the function; we can build this up in a picture.

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This is the Inverse Function Theorem: Let $f(x)$ be a function which is invertible and differentiable. For all x with $f'(f^{-1}(x)) \neq 0$, we have $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.
We can also get this via the chain rule: we know $x = f(f^{-1}(x))$. Just differentiate both sides! And we get $1 = f'(f^{-1}(x))(f^{-1})'(x)$, which is what we want.

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The inverse function theorem is very powerful: it lets us extend the power rule to all rational numbers instead of just integers, and find the derivatives of inverse trig functions.

Example

Let's compute $\frac{\mathrm{d}}{\mathrm{dx}}x^{1/3}$. Well if $g(x) = x^{1/3}$, then the inverse function is $f(x) = x^3$. So $\frac{\mathrm{d}}{\mathrm{dx}}x^{1/3} = \frac{1}{f'(g(x))} = \frac{1}{3g(x)^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$.

Content

The Extended Power Rule: Two parts. One: if n is a positive integer, $\frac{\mathrm{d}}{\mathrm{dx}}x^{1/n} = 1/nx^{1/n-1}$. Two: if n is a positive integer and m is an arbitrary integer, we have $\frac{\mathrm{d}}{\mathrm{dx}}x^{m/n} = m/nx^{m/n-1}$.

Let's prove this. Part one: If $g(x) = x^{1/n}$, then the inverse function is $f(x) = x^n$. Then $f'(x) = nx^{n-1}$ and $f'(g(x)) = n(x^{1/n})^{n-1} = nx^{1-1/n}$. Finally, we have $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{nx^{1-1/n}} = \frac{1}{n}x^{-(1-1/n)} = \frac{1}{n}x^{1/n-1}$.

Part two: If $g(x) = x^{m/n}$, then $g(x) = (x^{1/n})^m$, which we can differentiate with the chain rule. So $g'(x) = m(x^{1/n})^{m-1} \cdot \frac{1}{n}x^{1/n-1} = mx^{m/n-1/n} \cdot \frac{1}{n}x^{n-1} = \frac{m}{n}x^{m/n-1}$.

Example

Let's compute the equation of the tangent line to $f(x) = x^{2/3}$ at $x = 8$.

A point on the line is $(8, f(8)) = (8, 4)$, and $f'(x) = 2/3x^{-1/3}$, so $f'(8) = 2/3(8)^{-1/3} = 1/3$. Thus we have that $y - 4 = 1/3(x - 8)$.