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Contents

I. Big Idea Of The Day

Remarks

We can compute derivatives of polynomial and rational functions without using the limit definition, and trig functions, and sums, differences, products, quotients and compositions of these. Let's compute derivatives of inverse functions, and extend our power rule.

II. Summary of What We Can Differentiate

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- **Constant rule**: if c is some constant real number, and $f(x) = c$, then $f'(x) = 0$.
- **Power rule**: if n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- **Sum and Difference rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) = f'(x) \pm g'(x)$
- **Product rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)g(x)) = f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x)) + \frac{\mathrm{d}}{\mathrm{dx}}(f(x))g(x) = f(x)g'(x) + f'(x)g(x)$
- **Quotient rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{dx}}(f(x)) - f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x))}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
- **Chain rule**: If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$.

We also learned that $\frac{\mathrm{d}}{\mathrm{dx}}\sin(x) = \cos(x)$ and $\frac{\mathrm{d}}{\mathrm{dx}}\cos(x) = -\sin(x)$, so we can compute the derivatives of all trigonometric functions using the quotient rule (or can memorize them!). For posterity:

- $\frac{\mathrm{d}}{\mathrm{dx}}(\tan(x)) = \sec^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\cot(x)) = -\csc^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\sec(x)) = \sec(x)\tan(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\csc(x)) = -\csc(x)\cot(x)$

III. Derivatives of Inverse Functions

Remarks

Derivatives are slopes of tangent lines. If we take a function $f(x)$, then the slope of the tangent line at a $f'(a)$ should be very related to the slope of the tangent line of the inverse function, because the inverse function is just a reflection over the line $y = x$. As a matter of fact, the slope of the tangent line to the inverse function is the reciprocal of the slope of the tangent line to the function; we can build this up in a picture.

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Let's try to figure out the inverse function theorem. Our goal is to find the slope of the tangent line to f^{-1} at the point $x = a$, or $(f^{-1})'(a)$. The point on the graph is $(a, f^{-1}(a))$, which corresponds to the point $(f^{-1}(a), a)$ on the graph of f, as we just interchange the role of the x-coordinate and the y-coordinate. We get then, by the reciprocal relationship above, that $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.

This is the Inverse Function Theorem: Let $f(x)$ be a function which is invertible and differentiable. For all x with $f'(f^{-1}(x)) \neq 0$, we have $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

We can also get this via the chain rule: we know $x = f(f^{-1}(x))$. Just differentiate both sides! And we get $1 = f'(f^{-1}(x))(f^{-1})'(x)$, which is what we want.

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The inverse function theorem is very powerful: it lets us extend the power rule to all rational numbers instead of just integers, and find the derivatives of inverse trig functions.

Example

Let's compute $\frac{\mathrm{d}}{\mathrm{dx}} x^{1/3}$. Well if $g(x) = x^{1/3}$, then the inverse function is $f(x) = x^3$. So $\frac{\mathrm{d}}{\mathrm{dx}} x^{1/3} = \frac{1}{f'(g(x))} = \frac{1}{3g(x)^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$.

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The Extended Power Rule: Two parts. One: if n is a positive integer, $\frac{\mathrm{d}}{\mathrm{dx}}x^{1/n} = 1/nx^{1/n-1}$. Two: if n is a positive integer and m is an arbitrary integer, we have $\frac{\mathrm{d}}{\mathrm{dx}}x^{m/n} = m/nx^{m/n-1}$.
Let's prove this. Part one: If $g(x) = x^{1/n}$, then the inverse function is $f(x) = x^n$. Then $f'(x) = nx^{n-1}$ and $f'(g(x)) = n(x^{1/n})^{n-1} = nx^{1-1/n}$. Finally, we have $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{nx^{1-1/n}} = \frac{1}{n}x^{-(1-1/n)} = \frac{1}{n}x^{1/n-1}$.
Part two: If $g(x) = x^{m/n}$, then $g(x) = (x^{1/n})^m$, which we can differentiate with the chain rule. So $g'(x) = m(x^{1/n})^{m-1} \cdot \frac{1}{n}x^{1/n-1} = mx^{m/n-1/n} \cdot \frac{1}{n}x^{n-1} = \frac{m}{n}x^{m/n-1}$.

Example

Let's compute the equation of the tangent line to $f(x) = x^{2/3}$ at $x = 8$.
A point on the line is $(8, f(8)) = (8, 4)$, and $f'(x) = 2/3x^{-1/3}$, so $f'(8) = 2/3(8)^{-1/3} = 1/3$. Thus we have that $y - 4 = 1/3(x - 8)$.

IV. Derivatives of Inverse Trig Functions

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We can use the Inverse Function Theorem to find the derivatives of the inverse trig functions. In principle, it is pretty clear what to do; for example if $f(x) = \sin^{-1}(x)$, then $g(x) = \sin(x)$ is the inverse, so $f'(x) = \frac{1}{g'(f(x))} = \frac{1}{\cos(\sin^{-1}(x))}$. The last step is the potentially confusing one, where we try to figure out what $\cos(\sin^{-1}(x))$ is. We have done this before, just by drawing out the right triangle with angle $\theta = \sin^{-1}(x)$. We see if x is the ratio of the opposite to the hypotenuse, the adjacent side has length $\sqrt{1-x^2}$, so $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$, and $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

Example

Let's find the derivative of $f(x) = \cos^{-1}(x)$. The inverse function theorem says $f'(x) = \frac{1}{-\sin(\cos^{-1}(x))}$, which we need to figure out on a right triangle is $f'(x) = \frac{-1}{\sqrt{1-x^2}}$.

Let's find the derivative of $f(x) = \tan^{-1}(x)$. The inverse function theorem says $f'(x) = \frac{1}{\sec^2(\tan^{-1}(x))}$. We can use the right triangle that this is $f'(x) = \frac{1}{\frac{1}{(\cos(\tan^{-1}(x)))^2}} = (\cos(\tan^{-1}(x)))^2 = (\frac{1}{\sqrt{1+x^2}})^2 = \frac{1}{1+x^2}$.

Let's find the derivative of $f(x) = x^2 \sin^{-1}(\frac{1}{x})$. This requires product and chain rules: $f'(x) = 2x \sin^{-1}(\frac{1}{x}) + \frac{x^2}{\sqrt{1-(\frac{1}{x})^2}}(-\frac{1}{x^2})$.

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Let's just record the derivatives of inverse trig functions for posterity:

- $\frac{\mathrm{d}}{\mathrm{dx}} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{dx}} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{dx}} \tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{\mathrm{d}}{\mathrm{dx}} \cot^{-1}(x) = \frac{-1}{1+x^2}$
- $\frac{\mathrm{d}}{\mathrm{dx}} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{\mathrm{d}}{\mathrm{dx}} \csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$