MATH 1 FALL 2019 : LECTURE 24 FRI 11-08-19

SAMUEL TRIPP

Contents

I. Big Idea Of The Day 1
II. Summary of What We Can Differentiate 1
III. Implicit Differentiation 2
IV. Derivatives of Exponential and Log Functions 4

I. Big Idea Of The Day

Remarks

We can compute derivatives of polynomial and rational functions without using the limit definition, and trig functions, and sums, differences, products, quotients and compositions of these. Let's compute derivatives of inverse functions, and extend our power rule.

II. Summary of What We Can Differentiate

Content

- **Constant rule**: if c is some constant real number, and $f(x) = c$, then $f'(x) = 0$.
- **Power rule**: if n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- **Sum and Difference rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{dx}}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{dx}}(g(x)) = f'(x) \pm g'(x)$
- **Product rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(f(x)g(x)) = f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x)) + \frac{\mathrm{d}}{\mathrm{dx}}(f(x))g(x) = f(x)g'(x) + f'(x)g(x)$
- **Quotient rule**: $\frac{\mathrm{d}}{\mathrm{dx}}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{dx}}(f(x)) - f(x)\frac{\mathrm{d}}{\mathrm{dx}}(g(x))}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
- **Chain rule**: If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$.
- **Inverse Function Theorem**: $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a)}$

We also learned that $\frac{\mathrm{d}}{\mathrm{dx}}\sin(x) = \cos(x)$ and $\frac{\mathrm{d}}{\mathrm{dx}}\cos(x) = -\sin(x)$, so we can compute the derivatives of all trigonometric functions using the quotient rule (or can memorize them!). For posterity:

- $\frac{\mathrm{d}}{\mathrm{dx}}(\tan(x)) = \sec^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\cot(x)) = -\csc^2(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\sec(x)) = \sec(x)\tan(x)$
- $\frac{\mathrm{d}}{\mathrm{dx}}(\csc(x)) = -\csc(x)\cot(x)$

Finally, we have some derivatives of inverse trig functions:

- $\frac{\mathrm{d}}{\mathrm{dx}}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{dx}}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{dx}}\tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{\mathrm{d}}{\mathrm{dx}}\cot^{-1}(x) = \frac{-1}{1+x^2}$
- $\frac{\mathrm{d}}{\mathrm{dx}}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{\mathrm{d}}{\mathrm{dx}}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$

III. Implicit Differentiation

Content

Some equations clearly define y in terms of x. Other equations just have both variables in there, and implicitly define y in terms of x. Often times we can solve for y in terms of x, but not always. Look at the circle of radius 5, $x^2 + y^2 = 25$. We can kind of solve for y in terms of x, and get $y = \sqrt{25 - x^2}$, and differentiate this. If we do this, we get $\frac{\mathrm{d}}{\mathrm{dx}}y = -\frac{x}{\sqrt{25-x^2}}$.

On the other hand, we can do implicit differentiation. This just means differentiating the whole equation with respect to x, remembering everywhere that y is a function of x so we need to use the chain rule whenever we see it, and multiply by $\frac{\mathrm{dy}}{\mathrm{dx}}$.

Example

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{\mathrm{dy}}{\mathrm{dx}}$. Well we just differentiate both sides! And we get $\frac{\mathrm{d}}{\mathrm{dx}}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{dx}}(25)$, so $2x + 2y\frac{\mathrm{dy}}{\mathrm{dx}} = 0$, or $\frac{\mathrm{dy}}{\mathrm{dx}} = -x/y$. This matches up with what we have above, but works more generally! For the whole equation, not just the top half.

Example

Assuming that y is defined implicitly by the equation $\sin(y) = x$, find $\frac{\mathrm{dy}}{\mathrm{dx}}$. Differentiating both sides gives us $\cos(y)\frac{\mathrm{dy}}{\mathrm{dx}} = 1$, so $\frac{\mathrm{dy}}{\mathrm{dx}} = 1/\cos(y) = 1/\cos(\sin^{-1}(x)) = 1/\sqrt{1-x^2}$, which is good, because $y = \sin^{-1}(x)$, so we know that is the derivative!

Content

How to perform implicit differentiation:

(1) Differentiate both sides, taking care to use the chain rule any time you run into a y, so you get a $\frac{\mathrm{dy}}{\mathrm{dx}}$.

(2) Collect terms with $\frac{\mathrm{dy}}{\mathrm{dx}}$ on one side of the equals sign, and the terms without on the other side.

(3) Factor out $\frac{\mathrm{dy}}{\mathrm{dx}}$ from the appropriate side, and divide through by what is left to solve for $\frac{\mathrm{dy}}{\mathrm{dx}}$ as a fraction of functions.

Example

Let's find $\frac{\mathrm{dy}}{\mathrm{dx}}$ for $y\sin(xy) = y^2 + 2$. Well differentiating both sides gives $y\frac{\mathrm{d}}{\mathrm{dx}}\sin(xy) + \frac{\mathrm{dy}}{\mathrm{dx}}\sin(xy) = 2y\frac{\mathrm{dy}}{\mathrm{dx}}$. With the chain rule and product rule, we see $\frac{\mathrm{d}}{\mathrm{dx}}\sin(xy) = \cos(xy)(x\frac{\mathrm{dy}}{\mathrm{dx}}+y)$, so $y(\cos(xy)x\frac{\mathrm{dy}}{\mathrm{dx}} + \cos(xy)y) + \frac{\mathrm{dy}}{\mathrm{dx}}\sin(xy) = 2y\frac{\mathrm{dy}}{\mathrm{dx}}$. Collecting terms with $\frac{\mathrm{dy}}{\mathrm{dx}}$, we get $\cos(xy)y^2 = \frac{\mathrm{dy}}{\mathrm{dx}}(2y - \sin(xy) - xy\cos(xy))$, so $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\cos(xy)y^2}{2y - \sin(xy) - xy\cos(xy)}$.

Example

Let's find the equation of the tangent line to $xy^2 + \sin(\pi y) - 2x^2 = 10$ at the point $(2, -3)$.

First implicitly differentiate to get $2xy\frac{\text{dy}}{\text{dx}}+y^2+\cos(\pi y)\pi\frac{\text{dy}}{\text{dx}}-4x=0$, so $\frac{\text{dy}}{\text{dx}}(2xy+\cos(\pi y)\pi)=4x-y^2$, or $\frac{\text{dy}}{\text{dx}}=\frac{4x-y^2}{2xy+\cos(\pi y)\pi}$. Evaluating at the point $(2,-3)$ gives $\frac{\text{dy}}{\text{dx}}|_{(2,-3)}=\frac{-1}{-12+cos(-3\pi)}=\frac{-1}{-12-\pi}$. Then the equation of our tangent line is $y+3=\frac{1}{12+\pi}(x-2)$.

Example

We can do the same for $xy+\sin(x)=1$ at $(\pi/2,0)$.

IV. Derivatives of Exponential and Log Functions

Content

Consider the function $f(x)=e^x$. We want to figure out $\frac{\text{d}}{\text{dx}}e^x=f'(x)$. Using the limit definition, we get that $f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}=e^x\lim_{h\ to 0}\frac{e^h-1}{h}=e^xf'(0)$. We talked a long time ago about how one of the nice properties of e^x is that its derivative at 0 is 1, i.e. the slope of the tangent line at $x=0$ is 1. This means that $f'(0)=1$, and we get that $f'(x)=\frac{\text{d}}{\text{dx}}e^x=e^x$. This is a function whose derivative is the same function! It is the only function with this property.

Content

We know $f^{-1}(x)=\ln(x)$ is the inverse function to $f(x)=e^x$. Thus we have by the inverse function theorem that $\frac{\text{d}}{\text{dx}}\ln(x)=(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}=\frac{1}{f'(\ln x)}=\frac{1}{e^{\ln x}}=\frac{1}{x}$. This is a really nice use of the inverse function theorem to compute the derivative of $ln(x)$.

Content

We need to figure out the derivative of a general exponential function, $f(x)=b^x$ for $b>0$, $b\neq 1$. Well we know $f(x)=e^{\ln b^x}=e^{x\ln b}$, so by the chain rule, $f'(x)=e^{x\ln b}\frac{\text{d}}{\text{dx}}x\ln b=e^{x\ln b}\ln b=b^x\ln b$.