Week 3 - Inverse Functions

Friday, September 27, 2019 2:45 PM

Given a function f, the inverse function (if it exists) is written f^{-1} and has the property that

if $f(x)=y$, then $f^{-1}(y)=x$ for all x in f's domain.

e.g. $f(x)=x+5 \rightsquigarrow f^{-1}(x)=x-5$

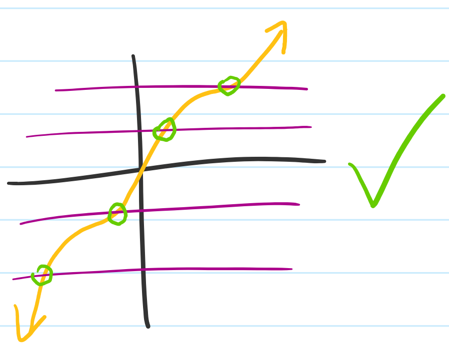
$f(x)=x^3-1 \rightsquigarrow f^{-1}(x)=\sqrt[3]{x+1}$

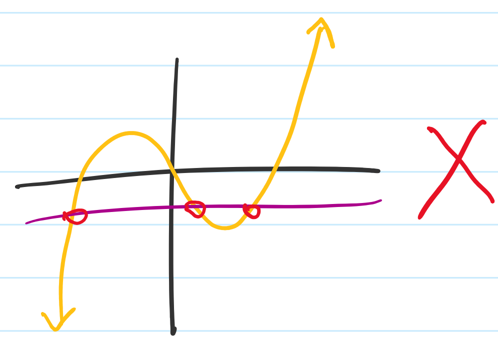
$f(x)=x^2 \rightsquigarrow \cancel{f^{-1}(x)=\pm\sqrt{x}}$ ← not a function!

When does a function have an inverse?

When it is one-to-one. ($f(x_1)\neq f(x_2)$ if $x_1\neq x_2$)

Equivalently, it passes the horizontal line test.





How to find $f^{-1}(x)$: $\boxed{\text{Solve } y=f(x) \text{ for } x.}$

e.g. $f(x)=3x-4 \rightsquigarrow y=3x-4$

e.g. $f(x) = 3x - 4 \rightsquigarrow y = 3x - 4$

$$y + 4 = 3x$$

$$x = \frac{y+4}{3} \rightsquigarrow f^{-1}(x) = \frac{x}{3} + \frac{4}{3}$$

Practice:

Find the inverses of: (if they exist)

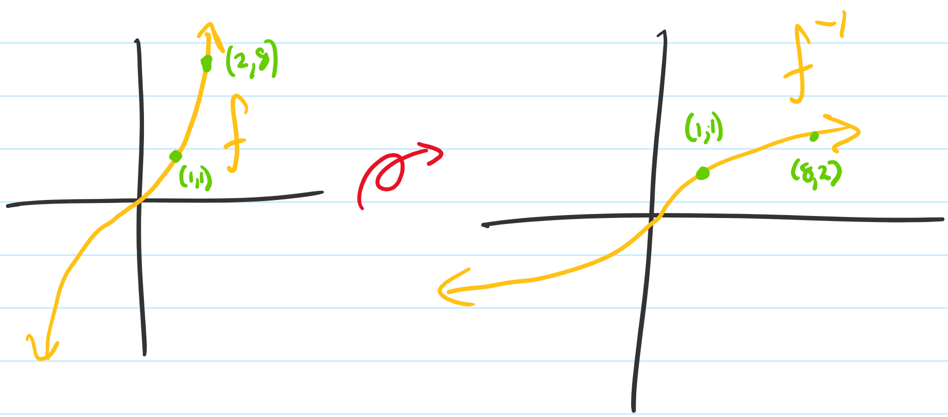
$$f(x) = \frac{3x}{x-2}$$

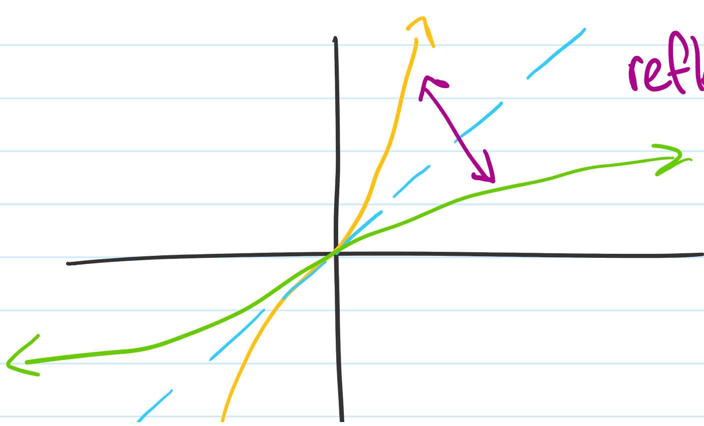
$$g(x) = \frac{1}{x}$$

$$h(x) = 5$$

Graphing

If the graph of $f(x)$ contains the point (x,y) then $f^{-1}(x)$ has the point (y,x)





reflection across the line $y = x$.

Restricting the domain

If f is not one-to-one (doesn't pass the horizontal line test) we can choose a smaller domain for f so that it does.

e.g. $f(x) = x^2$ passes the HLT on $[0, \infty)$

so $f^{-1}(x) = \sqrt{x}$ exists if we restrict the domain!

What is $f^{-1}(x)$ if we instead restrict $f(x) = x^2$ to $(-\infty, 0]$?

Trigonometric Functions

Does $\sin(x)$ have an inverse?

Not on $\mathbb{R}$, but we can restrict to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to give it one.

Similarly, restrict $\cos(x)$ to $[0, \pi]$ to get

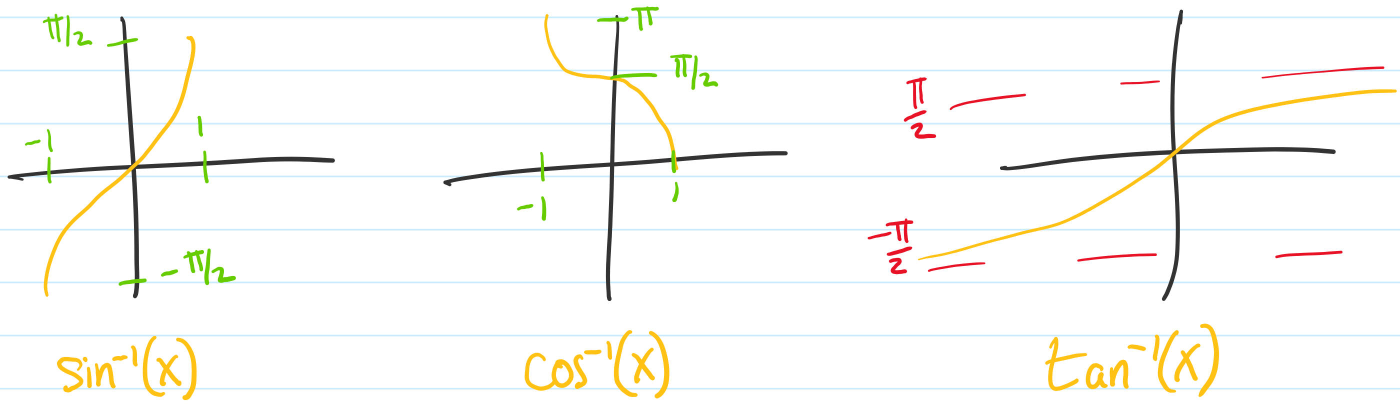
- $\sin^{-1}(x)$ (a.k.a. arcsin)
- $\cos^{-1}(x)$ (a.k.a. arccos)

Warning: $\sin^2 x = (\sin x)^2$, $\sin^{-1} x \neq \frac{1}{\sin x}$

Restrict $\tan(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to get

- $\tan^{-1}(x)$ (a.k.a. arctan)

Domains?



Examples

$f(x) = x^2 \rightsquigarrow f^{-1}(x) = \sqrt{x} \rightsquigarrow (f^{-1} \circ f)(x) = |x|$

$(f \circ f^{-1})(x) = x$

but domain $= [0, \infty)$

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$