Week 3 - Limit of a Sequence

Wednesday, October 2, 2019 3:21 PM

Given a sequence $\{a_n\}_{n=1}^{\infty}$, we can ask:

What happens to a_n as $n \to \infty$?

(Similar to behavior at ∞ for functions)

e.g.

$\{1+4n\} = 1, 5, 9, 13, \ldots \to \infty$ as $n \to \infty$

$\{1-(\frac{1}{2})^n\} = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots \to 1$ as $n \to \infty$

$\{(-1)^n\} = -1, 1, -1, 1, \ldots$ doesn't approach a value as $n \to \infty$

$\{(-1)^n/n\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots \to 0$ as $n \to \infty$

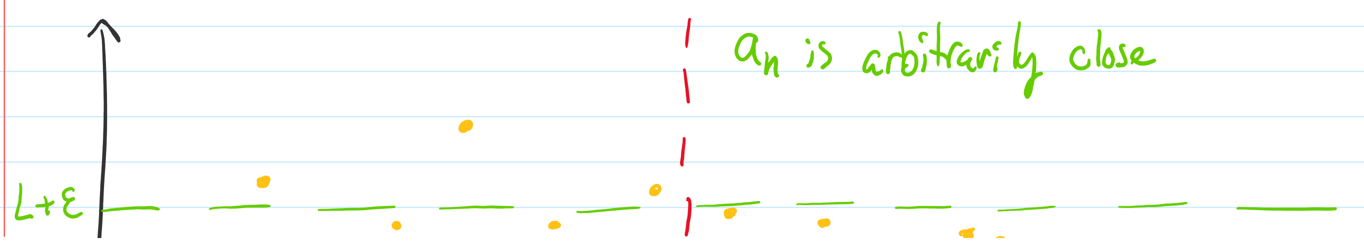
If a_n becomes arbitrarily close to some number L as n becomes sufficiently large, we say $\{a_n\}$ is a convergent sequence and L is the limit of the sequence.

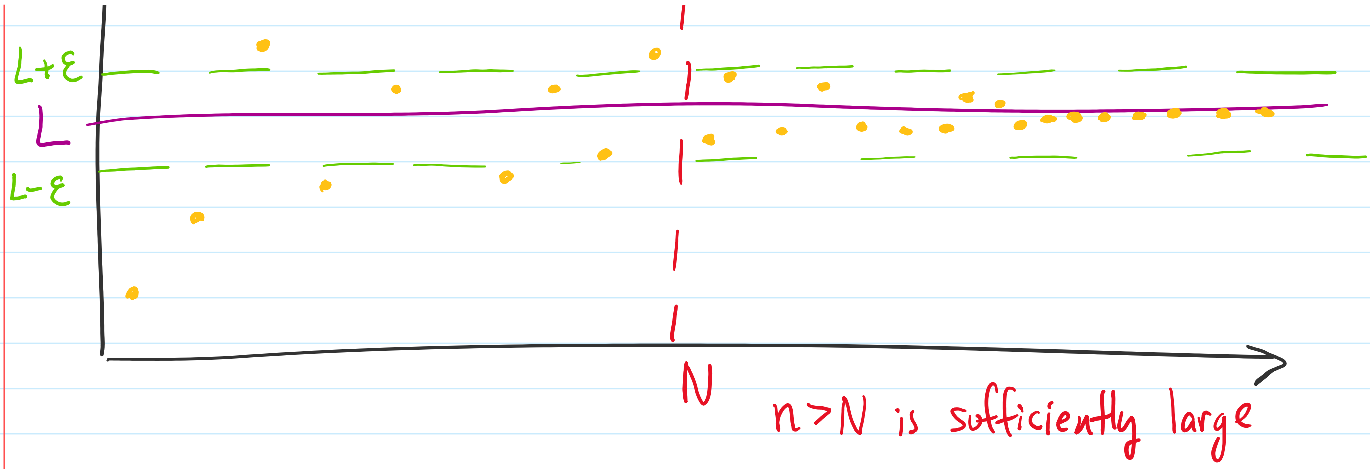
Written:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L$$

If $\{a_n\}$ is not convergent, we call it divergent.

Illustration





Same definition but more precise:

A sequence $\{a_n\}$ converges to L if for all $\varepsilon > 0$,
there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N$.

Remark: Adding a finite number of terms to the beginning of a sequence doesn't change the limit.

Example: The geometric sequence $\{r^n\}$:

$r^n \to 0$ if $r < 1$

$r^n \to 1$ if $r = 1$

$r^n \to \infty$ if $r > 1$

So... $(2/3)^n \to 0$ and $(1/4)^n \to 0$

what about $a_n = (2/3)^n + (1/4)^n$?

Since both $\to 0$, so does their sum!

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Limit Laws

If sequences $a_n \to A$ and $b_n \to B$, then for any real # c:

1) $\lim\limits_{n\to\infty} c = c$

2) $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n = cA$

3) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = A \pm B$

4) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\cdot\left(\lim\limits_{n\to\infty} b_n\right) = A\cdot B$

5) $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{A}{B}$ (as long as b_n is never 0 and $B \neq 0$.)

Fact: $\left\{\frac{1}{n}\right\} \to 0$

Q: What about $\left\{\frac{1}{n^2}\right\}$? A: $\frac{1}{n^2} = \frac{1}{n}\cdot\frac{1}{n}$, so $\frac{1}{n^2} \to 0\cdot 0 = \boxed{0.}$

Q: Find $\lim\limits_{n\to\infty} \dfrac{n}{n+1}$ A: $\dfrac{n}{n+1} = \dfrac{n(1)}{n\left(1+\frac{1}{n}\right)} = \dfrac{1}{1+\frac{1}{n}}$, $1 \to 1$, $1+\frac{1}{n} \to 1+0 = 1$, $\to \frac{1}{1} = \boxed{1.}$

Practice

Find $\lim\limits_{n\to\infty}$ (if it exists):

1) $4 - \dfrac{2}{n^4}$ $\to 4 - 0 = 4$

2) $\dfrac{n^2-n-3}{n^2+5} = \dfrac{1-\frac{1}{n}-\frac{3}{n^2}}{1+\frac{5}{n^2}} \to \dfrac{1}{1} = 1$

3) $n + \dfrac{1}{n}$ $\to \infty + 0 = \infty$

$1 + \frac{1}{n^2}$

3) $n + \frac{1}{n}$ $\to \infty + 0 = \infty$

4) $2^n - 2$ $\to \infty - 2 = \infty$

Squeeze Theorem

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$.

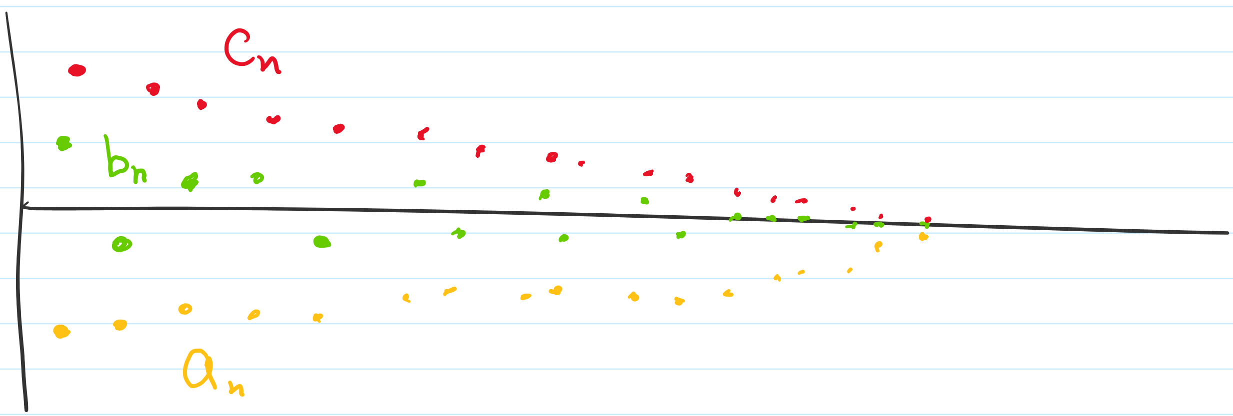
Suppose there is some integer N such that

$$a_n \le b_n \le c_n \quad \text{for all } n > N$$

If $a_n \to L$ and $c_n \to L$ for some $L \in \mathbb{R}$,

then $b_n \to L$ too.

Idea: If we can't figure $\lim\limits_{n\to\infty} b_n$ directly, we can "trap" it between two other sequences to figure out its limit.



Example: Find $\lim\limits_{n\to\infty} \frac{\cos n}{n}$

Since $\frac{-1}{n} \le \cos n \le \frac{1}{n}$, and $\frac{-1}{n}, \frac{1}{n} \to 0$,

$\frac{\cos n}{n} \to \boxed{0}$ as well

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Practice

Find $\lim_{n\to\infty}$ of:

1) $\left(-\frac{1}{3}\right)^n$ $\quad -\frac{1}{3^n} \le \quad \le \frac{1}{3^n} \to \boxed{0}$

2) $\frac{2n - \sin n}{n}$ $\quad 2 - \frac{\sin(n)}{n} \to \boxed{2}$