Week 5 - Theorems

Wednesday, October 16, 2019 2:23 PM

§2.3

The Squeeze Theorem (for functions)

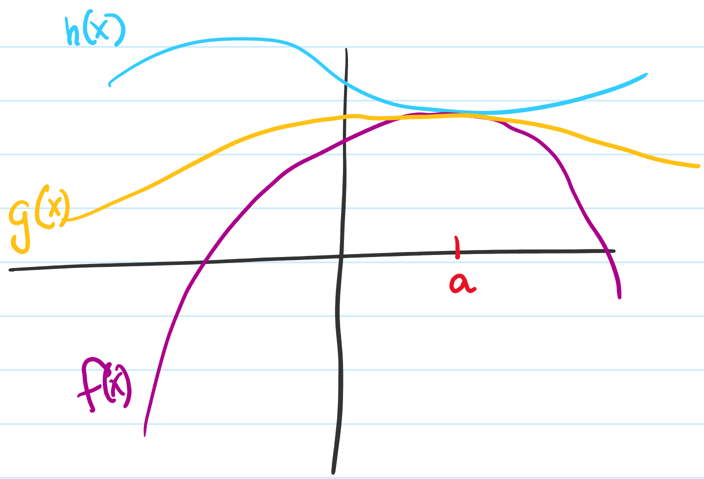
Let $f(x)$, $g(x)$, and $h(x)$ be functions.

If $f(x) \le g(x) \le h(x)$ for all $x \ne a$ in an open interval around a,

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then

$$\lim_{x \to a} g(x) = L.$$



e.g.

$\lim_{x \to 0} x\cos x$: Since $-1 \le \cos x \le 1$,

$-|x| \le x\cos x \le |x|$, and

$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$, so $\lim_{x \to 0} x\cos x = 0$ as well.

We already knew how to find $\lim_{x \to 0} x\cos x$ though: just plug in $x=0$!

Better example:

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$: Since $-1 \le \cos x \le 1$,

$x \to 0$

$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$, and since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$,

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ as well.

Recall: $f(x)$ is continuous at a if

1) $f(a)$ exists
2) $\lim_{x \to a} f(x)$ exists
3) $f(a) = \lim_{x \to a} f(x)$

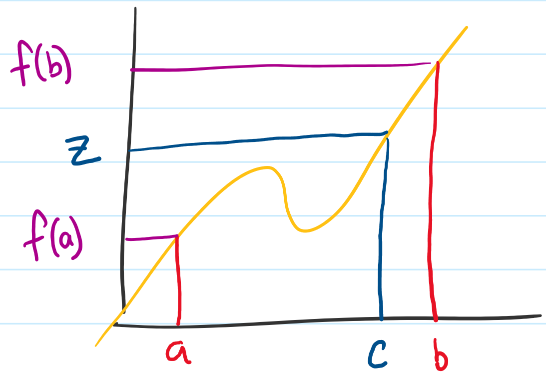
$f(x)$ continuous over $[a,b]$ if

- $f(x)$ continuous on (a,b)
- $f(x)$ left-continuous at b
- $f(x)$ right-continuous at a

The Intermediate Value Theorem

Let $f(x)$ be continuous over $[a,b]$.

If z is any real number between $f(a)$ and $f(b)$, then there is some c in $[a,b]$ such that $f(c) = z$.



e.g. Prove that $f(x) = x - \cos(x) = 0$ for some x.

$x=0$: $f(0) = 0 - \cos 0 = 0 - 1 = -1$

$x=\frac{\pi}{2}$: $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Since $z=0$ is between -1 and $\frac{\pi}{2}$, $x - \cos x = 0$ for some $x = c$ by the IVT.

When not to use IVT:

- $f(x) = \frac{1}{x}$. Since $f(-1) = -1$ and $f(1) = 1$, does that mean that $f(c) = 0$ for some $-1 < c < 1$?

Practice:

1) Find $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)$

2) Prove that $\theta = \cos\theta$ for some angle θ.