Week 7 - Derivatives of Trig Functions

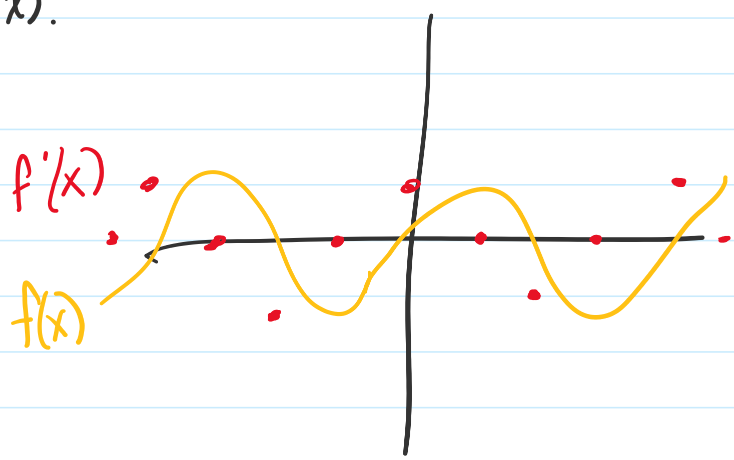
Tuesday, October 29, 2019 11:04 AM

We will need to know a couple important limits for todays derivatives:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Previously, we observed that when $f(x) = \sin x$, $f'(x)$ looked like $\cos(x)$.

Today, we'll figure out why.



Derivative of sin(x):

$$\frac{d}{dx}\left(\sin(x)\right) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\sin(h)\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\left(\cos(h) - 1\right)}{} + \lim_{h\to 0} \frac{\sin(h)\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\sin(h)\cos(x)}{h}$$

$$= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

Derivative of $\cos(x)$:

$$\frac{d}{dx}(\cos(x)) = \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \lim_{h\to 0} \cos(x)\cdot\frac{\cos(h)-1}{h} - \lim_{h\to 0} \sin(x)\cdot\frac{\sin(h)}{h}$$

$$= \cos(x)\cdot 0 - \sin(x)\cdot 1$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Q: What is $\frac{d}{dx}(-\sin(x))$?

$\sin(x) \to \cos(x) \to -\sin(x) \to -\cos(x) \to \sin(x) \to \ldots$

Q: If $f(x) = \sin(x)$ what is $f'''(x)$?

Q: If $f(x) = \sin(x)$, what is $f''''(x)$?

$f''''(x) = \sin(x)$.

Practice: Find the derivative.

1) $f(x) = 3\sin(x)$ $\quad f'(x) = 3\cos(x)$

2) $g(x) = x^2\cos(x)$ $\quad g'(x) = 2x\cos(x) - x^2\sin(x)$

3) $h(x) = \frac{1}{\sin(x)}$ $\quad h'(x) = \frac{0 \cdot \sin(x) - 1 \cdot \cos(x)}{\sin(x)^2} = \frac{-\cos(x)}{\sin^2(x)}$

$$= -\cot(x)\csc(x)$$

Derivatives of Other Trig Functions:

- We just computed $\frac{d}{dx}(\csc(x)) = -\cot(x)\csc(x)$

The rest of the trig functions:

- $$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot -\sin(x)}{(\cos(x))^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

- $$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \frac{0 \cdot \cos(x) - 1 \cdot -\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)}$$

$$\frac{d}{dx}(\sec(x)) = \tan(x)\sec(x)$$

$$\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)}$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Alternative way:

$$\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{1}{\tan(x)}\right) = \frac{0\cdot\tan(x) - 1\cdot\sec^2(x)}{\tan^2(x)} = \frac{-\sec^2(x)}{\tan^2(x)}$$

$$= -\frac{\frac{1}{\cos^2(x)}}{\frac{\sin^2(x)}{\cos^2(x)}} = -\frac{1}{\sin^2(x)} = -\csc^2(x)$$

Practice:

1) Find the derivative of $f(x) = \tan(x) - \cot(x)$

2) Find the equation of the tangent line to $g(x) = 2\sec(x)$ at $x = \frac{\pi}{4}$.

3) If $h(x) = \sin(x) + \cos(x)$, find $h^{(61)}(x)$ ← the 61st derivative of $h(x)$

1) $f'(x) = \sec^2(x) - (-\csc^2(x)) = \sec^2(x) + \csc^2(x)$

2) $g'(x) = 2(\tan(x)\sec(x))$, so

$g'\left(\frac{\pi}{4}\right) = 2\tan\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) = 2 \cdot 1 \cdot \frac{1}{\frac{\sqrt{2}}{2}} = 2 \cdot \frac{2}{\sqrt{2}} = 2\sqrt{2}$

$g\left(\frac{\pi}{4}\right) = 2 \cdot \frac{1}{\frac{\sqrt{2}}{2}} = 2\sqrt{2}$

$y - 2\sqrt{2} = 2\sqrt{2}\left(x - \frac{\pi}{4}\right) \rightsquigarrow y = 2\sqrt{2}\left(x - \frac{\pi}{4}\right) + 2\sqrt{2}$

3) $h^{(61)}(x) = h'(x) = \cos(x) - \sin(x)$