Week 8 - The Chain Rule

Tuesday, October 29, 2019 3:16 PM

Given two functions f, g, how do we differentiate their composition $f \circ g$?

e.g. $\sin(x^2)$ or $\sqrt{x+5}$

The Chain Rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

Proof:

$$\frac{d}{dx}\left(f(g(a))\right) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

What are we assuming here?

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x-a}$$

Let
$u = g(a)$
$v = g(x)$

$$= \lim_{v \to u} \frac{f(v) - f(u)}{v-u} \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x-a}$$

$$= f'(u) \cdot g'(a)$$

$$= f'(g(a)) \cdot g'(a)$$

Examples

$$\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

$$\frac{d}{dx}\left[\sin(x^2)\right] = \cos(x^2)\cdot 2x = 2x\cos(x^2)$$

$f(x) = \sin(x)$, $f'(x) = \cos(x)$
$g(x) = x^2$, $g'(x) = 2x$

$$\frac{d}{dx}\left(\sqrt{x+5}\right) = \frac{1}{2}(x+5)^{-\frac{1}{2}}\cdot 1 = \frac{1}{2\sqrt{x+5}}$$

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$
$g(x) = x+5$, $g'(x) = 1$

Leibniz Notation

Let $u = g(x)$, $y = f(u) = f(g(x))$.

Then we can write the chain rule $y' = f'(g(x))\cdot g'(x)$

as

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Pros: Easier to remember, looks like fractions

Cons: Not actually fractions

More Examples

$$\frac{d}{dx}\left(\frac{1}{(x^2+1)^2}\right) = -2(x^2+1)^{-3}\cdot 2x = \frac{-4x}{(x^2+1)^3}$$

$f(x) = x^{-2}$, $f'(x) = -2x^{-3}$
$g(x) = x^2+1$, $g'(x) = 2x$

$$\frac{d}{dx}\left(\tan(\cos(x))\right) = \sec^2(\cos(x)) \cdot -\sin(x) = \underline{-\sin(x)\sec^2(\cos(x))}$$

$f(x) = \tan(x)$, $f'(x) = \sec^2(x)$

$g(x) = \cos(x)$, $g'(x) = -\sin(x)$

What if we need to differentiate

$$K(x) = p(q(r(x))) \quad ?$$

Break up into two pieces at a time!

$$f(x) = p(x) \rightsquigarrow f'(x) = p'(x)$$

$$g(x) = q(r(x)) \rightsquigarrow g'(x) = q'(r(x)) \cdot r'(x)$$

so

$$K'(x) = p'(q(r(x))) \cdot q'(r(x)) \cdot r'(x)$$

Leibniz: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

for $u = q(r(x)) = q(v)$, $v = r(x)$