Week 8 - Derivatives of Inverse Functions

Tuesday, October 29, 2019 4:03 PM

Simplify the following:

a) $\left(\sqrt{x}\right)^2$ b) $\tan(\arctan(x))$ c) $(x-1)+1$ d) $e^{\ln x}$

All of the above are of the form $f(f^{-1}(x))$, and they're all equal to x!

We're going to use this rule ($f(f^{-1}(x)) = x$) to figure out a general formula for derivatives of inverse functions.

Start by taking the derivative of both sides:

$$\frac{d}{dx}\left(f(f^{-1}(x))\right) = f'(f^{-1}(x)) \cdot (f^{-1})'(x)$$

$$\frac{d}{dx}(x) = 1$$

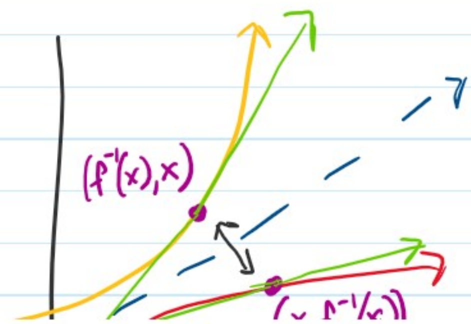
So $f'(f^{-1}(x))\,(f^{-1})'(x) = 1$ means

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

This is called the Inverse Function Theorem.

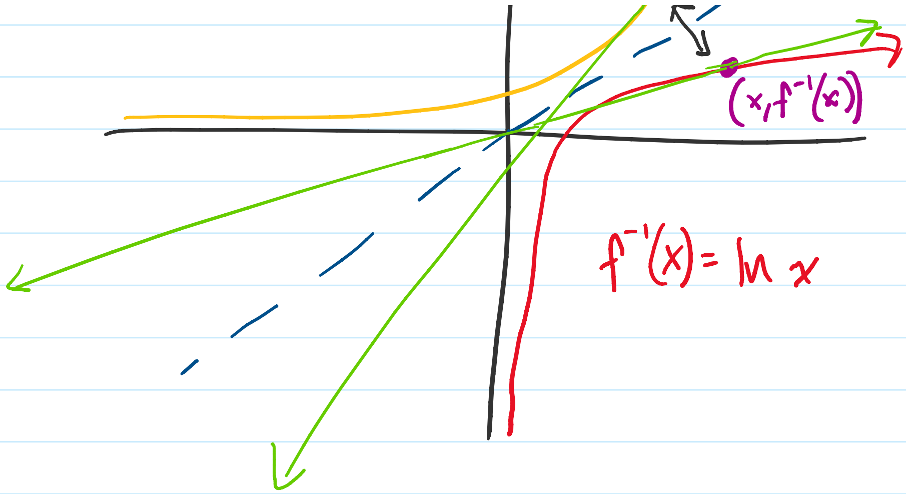
Graphically:

$f^{-1}(x)$ is obtained from $f(x)$ by reflecting across $y=x$ $f(x)=e^x$



their tangent lines are

their tangent lines are related by the same reflection!



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e.g. $\frac{d}{dx}(\sqrt{x}) = ?$

$f(x) = x^2 \qquad f'(x) = 2x$

$f^{-1}(x) = \sqrt{x} \qquad$ so $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{2\sqrt{x}}$

$\frac{d}{dx}(\sqrt[3]{x}) = ?$

$f(x) = x^3 \qquad f'(x) = 3x^2$

$f^{-1}(x) = \sqrt[3]{x} \qquad$ so $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{3(\sqrt[3]{x})^2}$

ie. $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$, $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$, ...

Let's try to prove the general pattern.

$$\frac{d}{dx}\left(x^{\frac{p}{q}}\right) = \frac{d}{dx}\left((x^p)^{\frac{1}{q}}\right)$$

$f(x) = x^{\frac{1}{q}} \qquad f'(x) = ? = \frac{1}{q}x^{\frac{1}{q}-1}$

$g(x) = x^p \qquad g'(x) = px^{p-1}$

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$j(x) = x^q \qquad j'(x) = qx^{q-1}$

$j^{-1}(x) = x^{\frac{1}{q}} \qquad \text{so} \quad (j^{-1})'(x) = \dfrac{1}{q\left(x^{\frac{1}{q}}\right)^{q-1}} = \dfrac{1}{q} x^{\frac{1}{q}-1}$

so

$$\frac{d}{dx}\left(x^{\frac{p}{q}}\right) = \frac{1}{q}\left(x^p\right)^{\frac{1}{q}-1} \cdot px^{p-1}$$

$$= \frac{p}{q} \cdot x^{\frac{p}{q}-p} \cdot x^{p-1}$$

$$= \frac{p}{q} x^{\frac{p}{q}-p+p-1}$$

$$= \frac{p}{q} x^{\frac{p}{q}-1}$$

... basically, the power rule works for fractions too.

Inverse Trig Functions

Let's try to differentiate $\arcsin(x)$.

$$\frac{d}{dx}\left(\arcsin(x)\right) = \ ?$$

$f(x) = \sin(x) \qquad f'(x) = \cos(x)$

$f^{-1}(x) = \arcsin(x) \qquad \text{so} \quad (f^{-1})'(x) = \dfrac{1}{\cos(\arcsin(x))}$

Can we simplify this further?

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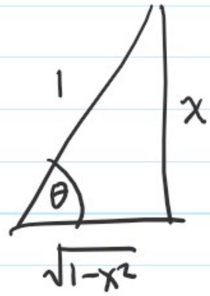
$\cos(\arcsin(x))$ is "the cosine of the angle with a sine of x"

We can draw a triangle:



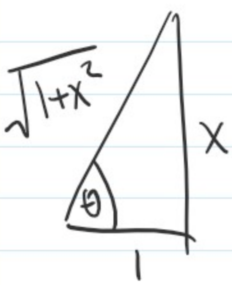
So since $\sin\theta = \frac{x}{1}$,

$$\cos(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

so $$(f^{-1})'(x) = \frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}(\arcsin(x))$$

Another example:

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2} = \frac{d}{dx}(\arctan x)$$



Practice:

1) Find $\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$

2) Find $\frac{d}{dx}(\operatorname{arccot}(x)) = \frac{-1}{1+x^2}$

$$\frac{d}{dx}(\operatorname{arcsec}(x)) = \frac{1}{\sec(\operatorname{arcsec} x)\cdot\tan(\operatorname{arcsec} x)} = \frac{1}{|x|\sqrt{x^2-1}} = \frac{d}{dx}(\operatorname{arcsec}(x))$$

$f(x) = \sec(x)$, $f^{-1}(x) = \sec(x)\tan(x)$

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