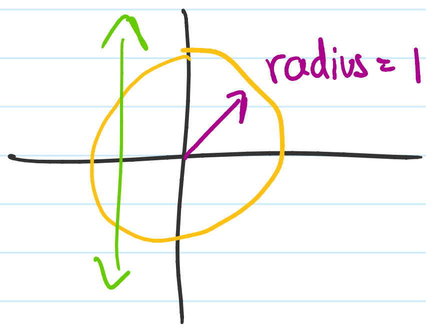
Week 8 - Implicit Differentiation

Wednesday, November 6, 2019 11:35 AM

$f'(x)$ tells us the slope of the tangent line to $y = f(x)$

How can we find the slope of the tangent line without solving for y? (Maybe y isn't even a function!)

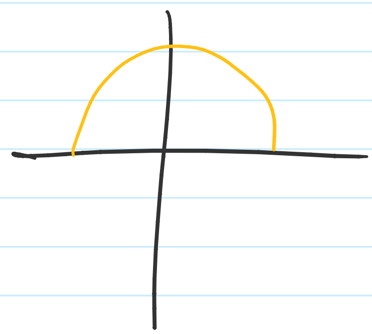
e.g. a circle



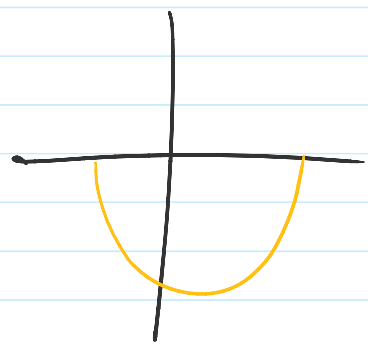
(fails vertical line test)

Option #1

Split into two functions



$y = \sqrt{1-x^2}$



$y = -\sqrt{1-x^2}$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{1}{2\sqrt{1-x^2}} \cdot -2x = \frac{-x}{\sqrt{1-x^2}}$$

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$

$g(x) = 1-x^2$, $g'(x) = -2x$

$$\frac{d}{dx}\left(-\sqrt{1-x^2}\right) = -\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{x}{\sqrt{1-x^2}}$$

Option #2 : Implicit Differentiation

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The circle is described by the equation $x^2+y^2=1$
we can "take the derivative of both sides" to get:

$$x^2+y^2=1$$

$$2x+2y\frac{dy}{dx}=0 \longrightarrow 2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

How does this compare to our previous answer?

$$y=\sqrt{1-x^2} \rightsquigarrow \frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}=\frac{-x}{y} \checkmark$$

$$y=-\sqrt{1-x^2} \rightsquigarrow \frac{dy}{dx}=\frac{x}{\sqrt{1-x^2}}=\frac{-x}{y} \checkmark$$

Note: Implicit differentiation gives us our slope in terms of x and y.

How to:

1) Take the derivative of both sides, treating y as a function of x.

e.g. $x=\tan y$

$$1=\sec^2 y\,\frac{dy}{dx}$$

2) Solve for $\frac{dy}{dx}$

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$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

3) Optional: Solve for y to write only in terms of x.

$y = \arctan(x)$, so

$$\frac{dy}{dx} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+x^2}$$

triangle method from last class

Practice: Find $\frac{dy}{dx}$, $\quad y^2 + y = x$

We can extend this techique to finding $\frac{d^2y}{dx^2}$ too:

$x^2+y^2=1$, $\frac{dy}{dx} = -\frac{x}{y}$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{y}\right) = \frac{(-1)(y)-\left(\frac{dy}{dx}\right)(-x)}{y^2} = \frac{-y+x\frac{dy}{dx}}{y^2}$$

$\| $

$\frac{d^2y}{dx^2}$

quotient rule!

$f=-x$, $f'=-1$

$g=y$, $g'=\frac{dy}{dx}$

$$= \frac{-y+x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y^2-x^2}{y^3} = -\frac{1}{y^3}$$

So $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$