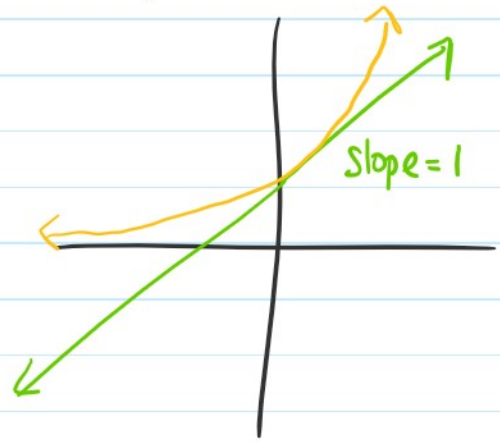
# Week 9 - Derivatives of Exponential Functions

Wednesday, November 6, 2019 12:00 PM

When we first introduced $e^x$, we said it had an interesting property: the slope of the tangent line to $e^x$ at $x=0$ was 1.




ie. $f(x)=e^x$, $f'(0)=1$

So $$\lim_{h\to 0}\frac{e^{0+h}-e^0}{h}$$

$$= \boxed{\lim_{h\to 0}\frac{e^h-1}{h} = 1}$$ ← We'll use this in a minute.

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Let's try to find the general formula for the derivative of $e^x$.

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

$$= \left(\lim_{h\to 0} e^x\right)\left(\lim_{h\to 0}\frac{e^h-1}{h}\right) = e^x\cdot 1 = e^x$$

So $\boxed{\frac{d}{dx}(e^x) = e^x}$ ($e^x$ is its own derivative!)

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Find the derivatives:

$f(x) = xe^x \rightsquigarrow f'(x) = e^x + xe^x$

$f(x) = xe^x \rightsquigarrow f'(x) = e^x + xe^x$

$g(x) = e^{2x} \rightsquigarrow g'(x) = 2e^{2x}$

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How can we find $\frac{d}{dx}(10^x)$ ?

$\rightarrow$ Rewrite as $\frac{d}{dx}\left(e^{x\ln(10)}\right) = \ln(10) \cdot e^{x\ln(10)} = 10^x \cdot \ln(10)$

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General Rule: $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$
(via chain rule)

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Now that we know $\frac{d}{dx}(e^x)$, we can use the Inverse function theorem to find $\frac{d}{dx}(\ln x)$ :

$f(x) = e^x$ , $f'(x) = e^x$

$f^{-1}(x) = \ln x$, $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

So

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

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General Rule : $\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$
(via chain rule)

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## Examples:

$$\frac{d}{dx}\left(\ln\left(x^2\right)\right) = \frac{1}{x^2}\cdot 2x = \frac{2x}{x^2} = \frac{2}{x}$$ (note: $\ln(x^2) = 2\ln(x)$)

$$\frac{d}{dx}\left(\ln(\sin(x))\right) = \frac{1}{\sin(x)}\cdot \cos(x) = \cot(x)$$

$$\frac{d}{dx}\left(\ln(\ln x)\right) = \frac{1}{\ln x}\cdot\frac{1}{x} = \frac{1}{x\ln(x)}$$

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How to find $\frac{d}{dx}(\log_{10} x)$ ?

Rewrite: $\log_{10} x = \frac{\ln x}{\ln 10}$, so $\frac{d}{dx}(\log_{10} x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 10}\right)$

$$= \frac{1}{\ln 10}\cdot\frac{1}{x} = \frac{1}{x\ln 10}$$

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**Logarithmic Differentiation** is a tool we can use to differentiate even more types of functions, like

$x^{\pi}$, $x^{x}$, $x^{\sin(x)}$, ... (basically, weird exponents)

Idea: 1) Start with a formula for $y$ in terms of $x$.

(can be explicit or implicit!)

2) Take ln of both sides

2) Take ln of both sides.

3) Take $\frac{d}{dx}$ of both sides and solve for $\frac{dy}{dx}$

(Implicit differentiation)

e.g. $f(x) = x^{\pi} \rightsquigarrow y = x^{\pi}$

$$\ln y = \ln\left(x^{\pi}\right) = \pi \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \pi\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \pi \frac{y}{x} = \pi\left(\frac{x^{\pi}}{x}\right) = \pi x^{\pi - 1}$$

Replacing $\pi$ by any constant $c$ proves the most general form of the power rule!

$$\frac{d}{dx}\left(x^{c}\right) = c x^{c-1}$$

e.g. $g(x) = x^{\sin(x)} \rightsquigarrow y = x^{\sin(x)}$

$$\ln y = \ln\left(x^{\sin x}\right) = \sin(x) \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \cos(x) \ln x + \frac{\sin(x)}{x}$$

$$\frac{dy}{dx} = y\left(\cos(x) \ln x + \frac{\sin(x)}{x}\right)$$

$$= x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$$

$$= x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$$