Week 9 - Extrema

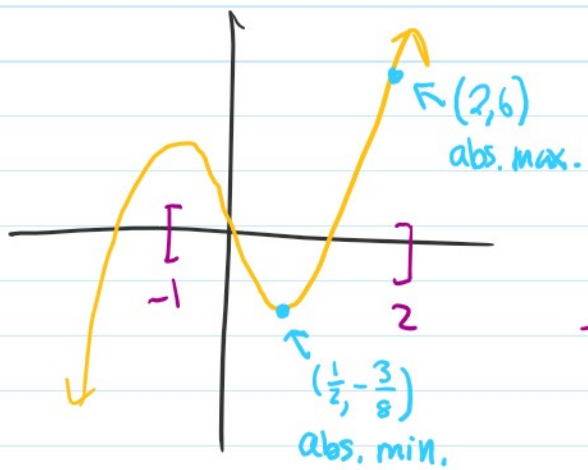
Monday, November 11, 2019 2:44 PM

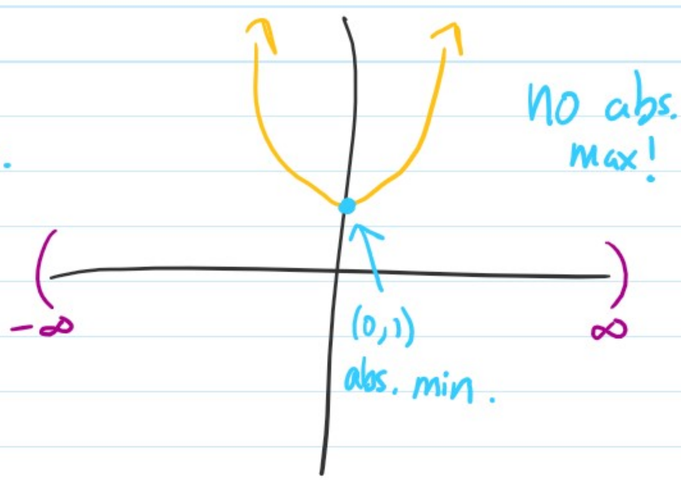
Note: "extrema" is the plural of "extremum", which is a word that can refer to a minimum or maximum.

Let f be a function on some interval I.

We say that f has an **absolute maximum** on I at c if $f(x) \le f(c)$ for all $x \in I$.

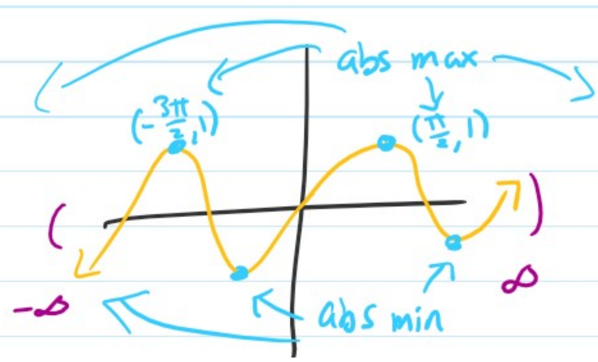
Similarly, we say that f has an **absolute minimum** on I at c if $f(x) \ge f(c)$ for all $x \in I$.





$f(x) = x^3 - x$, $I = [-1, 2]$

$f(x) = x^2 + 1$, $I = (-\infty, \infty)$



$f(x) = \sin(x), \; I = (-\infty, \infty)$

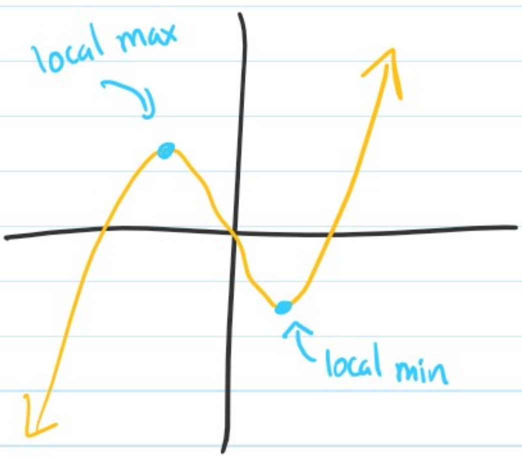
$f(x) = \dfrac{1}{x^2+1}, \; I = [-2, \infty)$

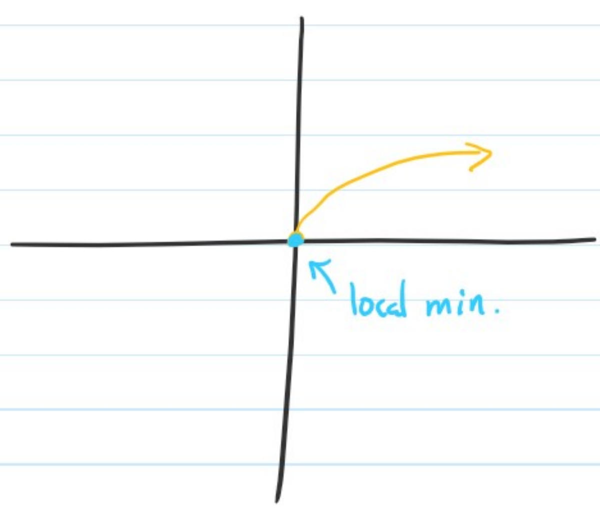
The Extreme Value Theorem says that any continuous function over a closed, bounded interval has both an absolute maximum and an absolute minimum.

A function f has a local maximum at c if
$f(c) \geq f(x)$ for all x in some open interval containing c.

Similarly, f has a local minimum at c if
$f(c) \leq f(x)$ for all x in some open interval containing c

e.g.





$f(x) = x^3 - x$

$f(x) = \sqrt{x}$

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Fermat's Theorem says that if f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

A point c at which $f'(c) = 0$ is called a critical point.

How do we know if a critical point is a local max, local min, or neither?

We can use the second-derivative test:

If $f'(c) = 0$ and

- $f''(c) < 0$, then c is a local max.
- $f''(c) > 0$, then c is a local min.

(if $f''(c) = 0$, the test is inconclusive.)

How can we use all this to help us find absolute extrema?

e.g. find the absolute max/min of $f(x) = \sin(x)$ on $[0, \frac{5\pi}{4}]$

Step 1: Find the critical points of f.

$f'(x) = \cos(x)$, so $\cos(x) = 0$ means $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

Step 2: Filter out the critical points not in the interval and add the endpoints to get all possible absolute extrema and add the endpoints to get all possible absolute extrema.

Possible extrema: $0, \frac{\pi}{2}, \frac{5\pi}{4}$

Step 3: Plug each value into $f(x)$.

largest output → absolute max

smallest output → absolute min

$f(0) = 0$

$f\left(\frac{\pi}{2}\right) = 1$ ← max!

$f\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ ← min!