Conditional Probability (cont...)

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Independent Events

Two events E and F are *independent* if both E and F have positive probability and if

$$P(E|F) = P(E) \ ,$$

and

$$P(F|E) = P(F) \ .$$

Theorem. *If $P(E) > 0$ and $P(F) > 0$, then E and F are independent if and only if*

$$P(E \cap F) = P(E)P(F) \ .$$

Example

Suppose that we have a coin which comes up heads with probability p, and tails with probability q. Now suppose that this coin is tossed twice. Let E be the event that heads turns up on the first toss and F the event that tails turns up on the second toss. Are these independent?

What if A is the event "the first toss is a head" and B is the event "the two outcomes are the same"?

What about I and J, where I is the event "heads on the first toss" and J is the event "two heads turn up."

A set of events $\{A_1,\ A_2,\ \ldots,\ A_n\}$ is said to be *mutually independent* if for any subset $\{A_i,\ A_j, \ldots,\ A_m\}$ of these events we have

$$P(A_i \cap A_j \cap \cdots \cap A_m) = P(A_i)P(A_j) \cdots P(A_m),$$

or equivalently, if for any sequence $\bar{A}_1$, $\bar{A}_2$, $\ldots$, $\bar{A}_n$ with $\bar{A}_j = A_j$ or $\tilde{A}_j$,

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) = P(\bar{A}_1)P(\bar{A}_2) \cdots P(\bar{A}_n).$$

Joint Distribution Functions

If we have several random variables $X_1, X_2, \ldots, X_n$ which correspond to a given experiment, then we can consider the joint random variable $\bar{X} = (X_1, X_2, \ldots, X_n)$ defined by taking an outcome ω of the experiment, and writing, as an n-tuple, the corresponding n outcomes for the random variables $X_1, X_2, \ldots, X_n$. Thus, if the random variable X_i has, as its set of possible outcomes the set R_i, then the set of possible outcomes of the joint random variable $\bar{X}$ is the Cartesian product of the R_i's, i.e., the set of all n-tuples of possible outcomes of the X_i's.

Definition

Let $X_1, X_2, \ldots, X_n$ be random variables associated with an experiment. Suppose that the sample space (i.e., the set of possible outcomes) of X_i is the set R_i. Then the joint random variable $\bar{X} = (X_1, X_2, \ldots, X_n)$ is defined to be the random variable whose outcomes consist of ordered n-tuples of outcomes, with the ith coordinate lying in the set R_i. The sample space Ω of $\bar{X}$ is the Cartesian product of the R_i's:

$$\Omega = R_1 \times R_1 \times \cdots \times R_n \ .$$

The joint distribution function of $\bar{X}$ is the function which gives the probability of each of the outcomes of $\bar{X}$.

Mutually Independent Variables

The random variables X_1, X_2, ..., X_n are mutually independent if

$$\begin{aligned} &P(X_1 = r_1, X_2 = r_2, \ldots, X_n = r_n) \\ &\qquad = P(X_1 = r_1)P(X_2 = r_2)\cdots P(X_n = r_n) \end{aligned}$$

for any choice of $r_1, r_2, \ldots, r_n$.

Thus, if $X_1,\ X_2, \ldots,\ X_n$ are mutually independent, then the joint distribution function of the random variable

$$\bar{X} = (X_1, X_2, \ldots, X_n)$$

is just the product of the individual distribution functions.

The distributions of the individual random variables are called *marginal distributions*.

Example

In a group of 60 people, the numbers who do or do not smoke and do or do not have cancer are reported as shown in Table 1.

	Not smoke	Smoke	Total
Not cancer	40	10	50
Cancer	7	3	10
Totals	47	13	60

Table 1: Smoking and cancer.

Let Ω be the sample space consisting of these 60 people. A person is chosen at random from the group. Let $C(\omega) = 1$ if this person has cancer and 0 if not, and $S(\omega) = 1$ if this person smokes and 0 if not. Then the joint distribution of $\{C, S\}$ is given in Table 2.

		S	
		0	1
C	0	40/60	10/60
	1	7/60	3/60

Table 2: Joint distribution.

- The marginal distributions of C and S are:

$$p_C = \begin{pmatrix} 0 & 1 \\ 50/60 & 10/60 \end{pmatrix},$$

$$p_S = \begin{pmatrix} 0 & 1 \\ 47/60 & 13/60 \end{pmatrix}.$$

- Are the random variables C and S independent?

Independent Trial Processes

A sequence of random variables X_1, X_2, ..., X_n that are mutually independent and that have the same distribution is called a sequence of independent trials or an *independent trials process*.

Example

We have a single experiment with sample space $R = \{r_1, r_2, \ldots, r_s\}$ and a distribution function

$$m_X = \begin{pmatrix} r_1 & r_2 & \ldots & r_n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} .$$

We repeat this experiment n times. To describe this total experiment, we choose as sample space the space

$$\Omega = R \times R \times \cdots \times R.$$

- We assign a distribution function to be the product distribution

$$m(\omega) = m(\omega_1) \cdot \ \ldots \ \cdot m(\omega_n) \ ,$$

with $m(\omega_j) = p_k$ when $\omega_j = r_k$.

- If we let X_j denote the jth coordinate of the outcome $(r_1, r_2, \ldots, r_n)$, the random variables X_1, ..., X_n form an independent trials process.

Example: Bernoulli Trials

Consider next a Bernoulli trials process with probability p for success on each experiment.

Let $X_j(\omega) = 1$ if the jth outcome is success and $X_j(\omega) = 0$ if it is a failure. Then X_1, X_2, ..., X_n is an independent trials process.

Each X_j has the same distribution function

$$m_j = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix},$$

where $q = 1 - p$.

If $S_n = X_1 + X_2 + \cdots + X_n$, then

$$P(S_n = j) = \binom{n}{j} p^j q^{n-j} \ .$$

Problems

A coin is tossed three times. Consider the following events

A: Heads on the first toss.

B: Tails on the second.

C: Heads on the third toss.

D: All three outcomes the same (HHH or TTT).

E: Exactly one head turns up.

1. Which of the following pairs of these events are independent?
   (1) A, B
   (2) A, D
   (3) A, E
   (4) D, E

2. Which of the following triples of these events are independent?

(1) A, B, C
(2) A, B, D
(3) C, D, E

What is the probability that a family of two children has

1. two boys given that it has at least one boy?
2. two boys given that the first child is a boy?

Prove that if A and B are independent so are

1. A and $\tilde{B}$.
2. $\tilde{A}$ and $\tilde{B}$.

In London, half of the days have some rain. The weather forecaster is correct $2/3$ of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won't rain, are both equal to $2/3$. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability $1/3$. Find

1. the probability that Pickwick has no umbrella, given that it rains.

2. the probability that it doesn't rain, given that he brings his umbrella.

> Three gamblers, A, B and C, take 12 balls of which 4 are white and 8 black. They play with the rules that the drawer is blindfolded, A is to draw first, then B and then C, the winner to be the one who first draws a white ball. What is the ratio of their chances?

Solve this problem first assuming that the ball is replaced after drawing and then assume that the game is without replacement.