Variance

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Definition

Let X be a numerically valued random variable with expected value $\mu = E(X)$. Then the *variance* of X, denoted by $V(X)$, is

$$V(X) = E((X - \mu)^2) \ .$$

Standard Deviation

The *standard deviation* of X, denoted by $D(X)$, is $D(X) = \sqrt{V(X)}$. We often write σ for $D(X)$ and σ^2 for $V(X)$.

Example

Consider one roll of a die. Let X be the number that turns up.

$$\begin{aligned} \mu &= E(X) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) \\ &= \frac{7}{2}\ . \end{aligned}$$

x	$m(x)$	$(x - 7/2)^2$
1	1/6	25/4
2	1/6	9/4
3	1/6	1/4
4	1/6	1/4
5	1/6	9/4
6	1/6	25/4

$$\begin{aligned} V(X) &= \frac{1}{6}\left(\frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4}\right) \\ &= \frac{35}{12}\,, \end{aligned}$$

Calculation of Variance

Theorem. *If X is any random variable with $E(X) = \mu$, then*

$$V(X) = E(X^2) - \mu^2 \ .$$

Example (cont)

$$\begin{aligned} E(X^2) &= 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 9\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 25\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) \\ &= \frac{91}{6} \ , \end{aligned}$$

and,

$$V(X) = E(X^2) - \mu^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \ .$$

Properties of Variance

Theorem. *If X is any random variable and c is any constant, then*

$$V(cX) = c^2 V(X)$$

and

$$V(X + c) = V(X)\ .$$

Theorem. *Let X and Y be two independent random variables. Then*

$$V(X+Y) = V(X) + V(Y).$$

Theorem. *Let X_1, X_2, ..., X_n be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$. Let*

$$S_n = X_1 + X_2 + \cdots + X_n$$

be the sum, and

$$A_n = \frac{S_n}{n}$$

ou be the average. Then

$$\begin{aligned} E(S_n) &= n\mu \ , \\ V(S_n) &= n\sigma^2 \ , \\ E(A_n) &= \mu \ , \\ V(A_n) &= \frac{\sigma^2}{n} \ . \end{aligned}$$

Practice Problems

Four balls are drawn at random, without replacement, from an urn containing 4 red balls and 3 blue. Let X be the number of red balls drawn.

1. What is the range of X?

2. What is the probability that $X = 2$?

3. What is the probability that $X = 2$ if, each time a ball is drawn, it is replaced in the urn?

You deal yourself a hand of 4 cards from an ordinary 52-card deck.

1. What is the probability of getting one card for each suit?

2. What is the probability of getting 3 cards of one suit and one of another?

3. What is the probability of getting 2 cards of one suit and two of another?

You flip a coin fair 5 times. Let A be the event that you get at least 2 heads, B the event that you get an even number of heads.

1. Compute $P(A)$, and write it as a fraction.

2. Compute $P(B)$, and write it as a fraction.

3. Determine whether A and B are independent.

In a certain manufacturing process, the (Fahrenheit) temperature is a random variable F with distribution

$$P_F = \begin{pmatrix} 60 & 61 & 62 & 63 & 64 \\ 1/10 & 2/10 & 4/10 & 2/10 & 1/10 \end{pmatrix}.$$

1. Find $E(F)$ and $V(F)$.

2. Define $T = F - 62$. Find $E(T)$ and $V(T)$, and compare these answers with those in part (a).

3. It is decided to report the temperature readings on a Celsius scale, that is, $C = (5/9)(F - 32)$. What is the expected value and variance for the readings now?

1. In how many ways can the letters of the word ROTOR be arranged?

2. What if we must leave T in the middle?