Important Distributions

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Discrete Uniform Distribution

- All outcomes of an experiment are equally likely.

- If X is a random variable which represents the outcome of an experiment of this type, then we say that X is *uniformly distributed.*

- If the sample space S is of size n, where $0 < n < \infty$, then the distribution function $m(\omega)$ is defined to be $1/n$ for all $\omega \in S$.

Binomial Distribution

- The distribution of the random variable which counts the number of heads which occur when a coin is tossed n times, assuming that on any one toss, the probability that a head occurs is p.

- The distribution function is given by the formula

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k} \ ,$$

where $q = 1 - p$.

Exercise

A die is rolled until the first time T that a six turns up.

1. What is the probability distribution for T?

2. Find $P(T > 3)$.

3. Find $P(T > 6|T > 3)$.

Geometric Distribution

- Consider a Bernoulli trials process continued for an infinite number of trials; for example, a coin tossed an infinite sequence of times.

- Let T be the number of trials up to and including the first success. Then

$$\begin{aligned} P(T = 1) &= p \ , \\ P(T = 2) &= qp \ , \\ P(T = 3) &= q^2 p \ , \end{aligned}$$

and in general,

$$P(T = n) = q^{n-1} p \ .$$

Exercise

Cards are drawn, one at a time, from a standard deck; each card is replaced before the next one is drawn. Let X be the number of draws necessary to get an ace. Find $E(X)$.

Example

Suppose a line of customers waits for service at a counter. It is often assumed that, in each small time unit, either 0 or 1 new customers arrive at the counter. The probability that a customer arrives is p and that no customer arrives is $q = 1 - p$. Let T be the time until the next arrival What is the probability that no customer arrives in the next k time units, that is, for $P(T > k)$.

Negative Binomial Distribution

- Suppose we are given a coin which has probability p of coming up heads when it is tossed.

- We fix a positive integer k, and toss the coin until the kth head appears.

- Let X represent the number of tosses. When $k = 1$, X is geometrically distributed.

- For a general k, we say that X has a *negative binomial distribution*.

- What is the probability distribution $u(x, k, p)$ of X?

Example

A fair coin is tossed until the second time a head turns up. The distribution for the number of tosses is $u(x, 2, p)$. What is the probability that x tosses are needed to obtain two heads.

The Poisson Distribution

- The Poisson distribution can be viewed as arising from the binomial distribution, when n is large and p is small.

- The Poisson distribution with parameter λ is obtained as a limit of binomial distributions with parameters n and p, where it was assumed that $np = \lambda$, and $n \to \infty$.

$$P(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda} \ .$$

Example

- A typesetter makes, on the average, one mistake per 1000 words. Assume that he is setting a book with 100 words to a page.

- Let S_{100} be the number of mistakes that he makes on a single page.

- Then the exact probability distribution for S_{100} would be obtained by considering S_{100} as a result of 100 Bernoulli trials with $p = 1/1000$.

- The expected value of S_{100} is $\lambda = 100(1/1000) = .1$.

- The exact probability that $S_{100} = j$ is $b(100, 1/1000, j)$, and the Poisson approximation is

$$\frac{e^{-.1}(.1)^j}{j!}.$$

Exercise

The Poisson distribution with parameter $\lambda = .3$ has been assigned for the outcome of an experiment. Let X be the outcome function. Find $P(X = 0)$, $P(X = 1)$, and $P(X > 1)$.

Exercise

In a class of 80 students, the professor calls on 1 student chosen at random for a recitation in each class period. There are 32 class periods in a term.

1. Write a formula for the exact probability that a given student is called upon j times during the term.

2. Write a formula for the Poisson approximation for this probability. Using your formula estimate the probability that a given student is called upon more than twice.

Hypergeometric Distribution

- Suppose that we have a set of N balls, of which k are red and $N - k$ are blue.

- We choose n of these balls, without replacement, and define X to be the number of red balls in our sample.

- The distribution of X is called *the hypergeometric distribution*.

- Note that this distribution depends upon three parameters, namely N, k, and n.

- We will use the notation $h(N, k, n, x)$ to denote $P(X = x)$.

- The distribution function is

$$h(N, k, n, x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \ .$$

Example

A bridge deck has 52 cards with 13 cards in each of four suits: spades, hearts, diamonds, and clubs. A hand of 13 cards is dealt from a shuffled deck. Find the probability that the hand has

1. a distribution of suits 4, 4, 3, 2 (for example, four spades, four hearts, three diamonds, two clubs).

2. a distribution of suits 5, 3, 3, 2.