Law of Large Numbers

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- An intuitive way to view the probability of a certain outcome is the frequency with which that outcome occurs in the long run.

- We defined probability mathematically as a value of a distribution function for the random variable representing the experiment.

- The Law of Large Numbers shows that this model is consistent with the frequency interpretation of probability.

Chebyshev Inequality

Theorem. *Let X be a discrete random variable with expected value $\mu = E(X)$, and let $\epsilon > 0$ be any positive real number. Then*

$$P(|X - \mu| \geq \epsilon) \leq \frac{V(X)}{\epsilon^2} \ .$$

Proof. Let $m(x)$ denote the distribution function of X.

$$P(|X - \mu| \geq \epsilon) = \sum_{|x-\mu| \geq \epsilon} m(x) \ .$$

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$\square$

Example

- Let X by any random variable with $E(X) = \mu$ and $V(X) = \sigma^2$.

- Then, if $\epsilon = k\sigma$, Chebyshev's Inequality states that

$$P(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2} \ .$$

- Thus, for any random variable, the probability of a deviation from the mean of more than k standard deviations is $\leq 1/k^2$.

- Chebyshev's Inequality is the best possible inequality in the sense that, for any $\epsilon > 0$, it is possible to give an example of a random variable for which Chebyshev's Inequality is in fact an equality.

- Given $\epsilon > 0$, choose X with distribution

$$p_X = \begin{pmatrix} -\varepsilon & -\varepsilon \\ 1/2 & 1/2 \end{pmatrix} .$$

Then $E(X) = 0$, $V(X) = \epsilon^2$, and

$$P(|X - \mu| \geq \epsilon) = \frac{V(X)}{\epsilon^2} = 1 .$$

Law of Large Numbers

Theorem. *Let X_1, X_2, $\ldots$, X_n be an independent trials process, with finite expected value $\mu = E(X_j)$ and finite variance $\sigma^2 = V(X_j)$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for any $\epsilon > 0$,*

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0$$

as $n \to \infty$. Equivalently,

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \epsilon\right) \to 1$$

as $n \to \infty$.

Proof

- Since X_1, X_2, $\ldots$, X_n are independent and have the same distributions,

$$V(S_n) = n\sigma^2 \ ,$$

$$V(\frac{S_n}{n}) = \frac{\sigma^2}{n} \ .$$

$$E(\frac{S_n}{n}) = \mu \ .$$

- By Chebyshev's Inequality, for any $\epsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2} \ .$$

Law of Averages

- Consider the important special case of Bernoulli trials with probability p for success.

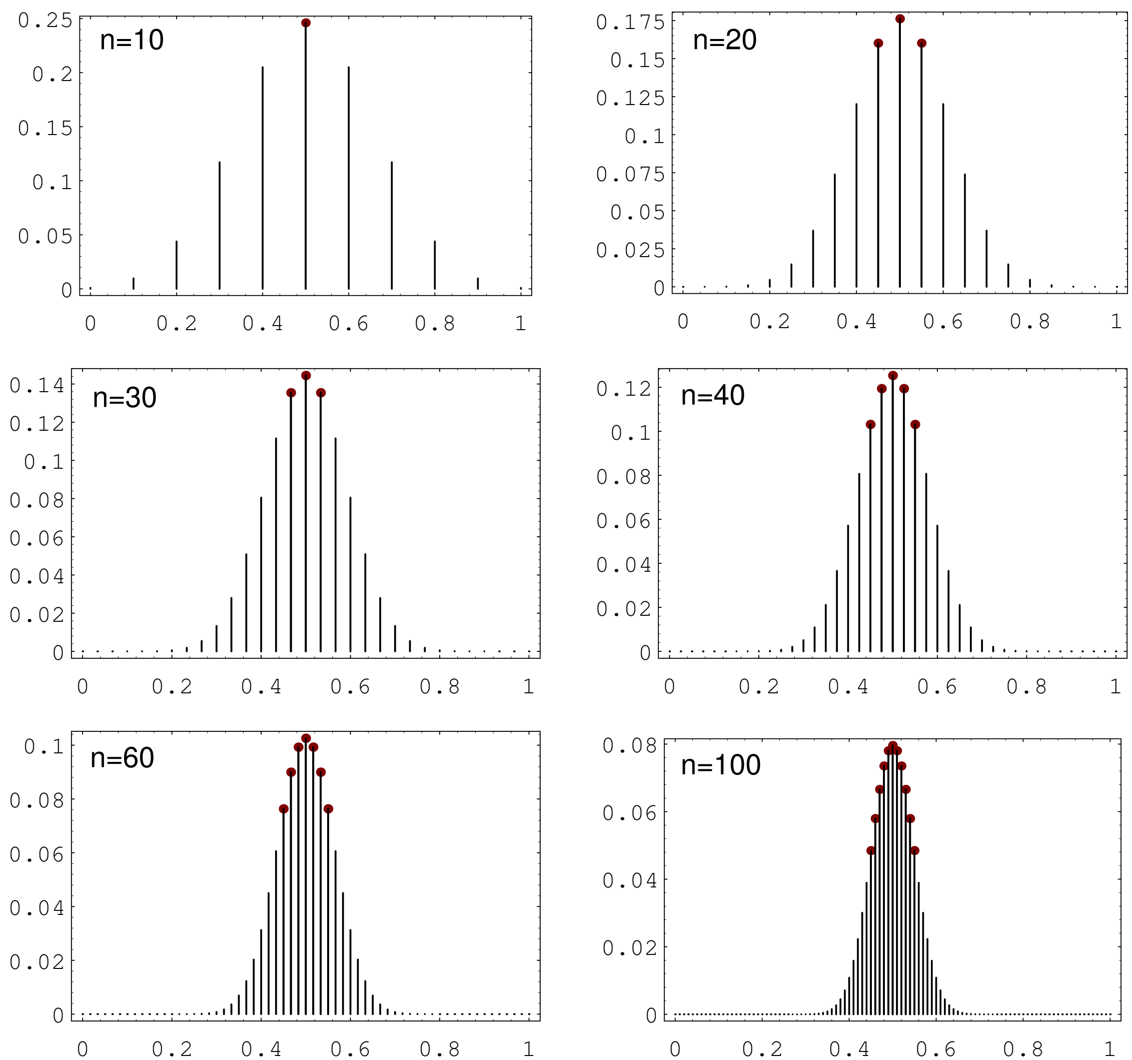
- Let $X_j = 1$ if the jth outcome is a success and 0 if it is a failure.

- Then $S_n = X_1 + X_2 + \cdots + X_n$ is the number of successes in n trials and $\mu = E(X_1) = p$.

- The Law of Large Numbers states that for any $\epsilon > 0$

$$P\left(\left|\frac{S_n}{n} - p\right| < \epsilon\right) \to 1$$

as $n \to \infty$.


n=10
n=20
n=30
n=40
n=60
n=100

Die Rolling

- Consider n rolls of a die. Let X_j be the outcome of the jth roll.
- Then $S_n = X_1 + X_2 + \cdots + X_n$ is the sum of the first n rolls.
- This is an independent trials process with $E(X_j) = 7/2$.
- Thus, for any $\epsilon > 0$

$$P\left(\left|\frac{S_n}{n} - \frac{7}{2}\right| \geq \epsilon\right) \to 0$$

as $n \to \infty$.

Problem

A fair coin is tossed 100 times. The expected number of heads is 50, and the standard deviation for the number of heads is $(100 \cdot 1/2 \cdot 1/2)^{1/2} = 5$. What does Chebyshev's Inequality tell you about the probability that the number of heads that turn up deviates from the expected number 50 by three or more standard deviations (i.e., by at least 15)?

Problem

Let X be a random variable with $E(X) = 0$ and $V(X) = 1$. What integer value k will assure us that $P(|X| \geq k) \leq .01$?

Problem

Let S_n be the number of successes in n Bernoulli trials with probability p for success on each trial. Find the maximum possible value for $p(1-p)$ if $0 < p < 1$. Using this result and the fact that for any $\epsilon > 0$

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \epsilon\right) \leq \frac{p(1-p)}{n\epsilon^2} \ ,$$

show that the estimate

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \epsilon\right) \leq \frac{1}{4n\epsilon^2}.$$

Problem

We have two coins: one is a fair coin and the other is a coin that produces heads with probability $3/4$. One of the two coins is picked at random, and this coin is tossed n times. Let S_n be the number of heads that turns up in these n tosses. Does the Law of Large Numbers allow us to predict the proportion of heads that will turn up in the long run? After we have observed a large number of tosses, can we tell which coin was chosen? How many tosses suffice to make us 95 percent sure?