MATH 20, WORKSHEET 3

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DUE FRIDAY SEPTEMBER 29

(1) You are given two boxes and an even number of balls. Half of the balls are white and half are black. You are asked to distribute the balls in the boxes with no restriction placed on the number of either type in an box. How should you distribute the balls in the boxes to maximize the probability of obtaining a white ball if an box is chosen at random and a ball drawn out at random?
Hints:

- Start by showing that you can do better than having the same number of white and black balls in each box.
- What are the odds if you put all the black balls in one box and all the white in the other box? If you move one black (or white) ball from one box to the other, what happens to your odds?

(2) A deck of playing cards can be described as a Cartesian product

$$\text{Deck} = \text{Suit} \times \text{Rank}\ ,$$

where $\text{Suit} = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and $\text{Rank} = \{2, 3, \ldots, 10, \text{J}, \text{Q}, \text{K}, \text{A}\}$. This just means that every card may be thought of as an ordered pair like $(\diamondsuit, 2)$. By a **suit event** we mean any event A contained in Deck which is described in terms of Suit alone. For instance, if A is "the suit is red," then

$$A = \{\diamondsuit, \heartsuit\} \times \text{Rank}\ ,$$

so that A consists of all cards of the form $(\diamondsuit, r)$ or $(\heartsuit, r)$ where r is any rank. Similarly, a **rank event** is any event described in terms of rank alone.

(a) Show that if A is any suit event and B any rank event, then A and B are independent. (We can express this briefly by saying that suit and rank are independent.)

(b) Throw away the ace of spades. Show that now no nontrivial (i.e., neither empty nor the whole space) suit event A is independent of any nontrivial rank event B. Hint: Here independence comes down to

$$c/51 = (a/51) \cdot (b/51) \ ,$$

where a, b, c are the respective sizes of A, B and $A \cap B$. It follows that 51 must divide ab, hence that 3 must divide one of a and b, and 17 the other. But the possible sizes for suit and rank events preclude this.

(c) Show that the deck in (b) nevertheless does have pairs A, B of nontrivial independent events. Hint: Find 2 events A and B of sizes 3 and 17, respectively, which intersect in a single point.

(d) Add a joker to a full deck. Show that now there is no pair A, B of nontrivial independent events. Hint: See the hint in (b); 53 is prime.