WorkSheet #8

1) $$p_{ij}^{(n)} := P(X_n = s_j \mid X_0 = s_i)$$

<u>Goal</u> Show that we can read $p_{ij}^{(n)}$ from the (i,j) - entry of the matrix P^n, i.e. $p_{ij}^{(n)} = (P^n)_{i,j}$

i) For $n=1$, P^1 is the transition matrix and by definition its entries are $p_{ij}^{(1)} = p_{ij}$

ii) Assume that the statement holds for $n=k$ and with that we will show that it also holds for $n=k+1$

$$p_{ij}^{(k+1)} = P(X_{k+1} = s_j \mid X_0 = s_i)$$

$$= \sum_{\ell=1}^{r} P(X_{k+1} = s_j, X_k = s_\ell \mid X_0 = s_i) \quad \left(\text{decomposing the event in a disjoint union of events}\right)$$

$$= \sum_{\ell=1}^{r} P(X_{k+1} = s_j \mid X_k = s_\ell, X_0 = s_i) P(X_k = s_\ell \mid X_0 = s_i) \quad \left(\begin{array}{c} P(A \cap B) \\ \parallel \\ P(A \mid B) P(B) \end{array}\right)$$

$$= \sum_{\ell=1}^{r} P(X_{k+1} = s_j \mid X_k = s_\ell) P(X_k = s_\ell \mid X_0 = s_i) \quad \left(\text{Markov property}\right)$$

$$= \sum_{\ell=1}^{r} p_{\ell j} \, p_{i\ell}^{(k)}$$

$$= \sum_{l=1}^{r} p_{lj} \left(P^{k}\right)_{i,l} \qquad \text{(since the claim holds for } n=k\text{)}$$

$$= \sum_{l=1}^{r} \left(P^{k}\right)_{i,l} \left(P\right)_{l,j}$$

$$= \left(P^{k+1}\right)_{i,j} \qquad \text{(by the matrix product formula)}$$

Thus, by induction we showed that the claim holds for all n

2)

a) $P(Y \geq k) = P\left(\begin{array}{l}\text{not being asorbed by the} \\ \text{kth step, given that it} \\ \text{started on the state } s_i\end{array}\right)$

$$= P\left(X_k \text{ is in a transient state} \mid X_0 = s_i\right)$$

$$= P\big(X_k \in \overbrace{\{s_1, \ldots, s_t\}}^{\text{all the transient states}} \mid X_0 = s_i\big)$$

$$= \sum_{l=1}^{t} \underbrace{P(X_k = s_l \mid X_0 = s_i)}$$

(i,l) entry of $P^k = \left[\begin{array}{c|c} Q^k & \text{something} \\ \hline & I \end{array}\right]$

$\parallel$

(i,l) entry of Q^k

$$= \sum_{l=1}^{t} (Q^k)_{i,l}$$

b)

$$E[Y] = \sum_{k \geq 0} P(Y \geq k) \qquad \text{by worksheet 4}$$

$$= \sum_{k=0}^{+\infty} \sum_{l=1}^{t} (Q^k)_{i,l}$$

$$= \sum_{l=1}^{t} \sum_{k=0}^{+\infty} (Q^k)_{i,l}$$

$$= \sum_{l=1}^{t} \underbrace{\left(I + Q + Q^2 + \cdots + Q^n + \cdots \right)}_{N}{}_{i,l}$$

$$= \sum_{l=1}^{t} (N)_{i,l} = \sum_{l=1}^{t} n_{il}$$

= sum of ith row entries of N.

3) 5

$$P\left(\text{probability of being absorbed by state } j \;\middle|\; X_0 = s_i\right)$$

$$\|$$

$$\sum_{k=1}^{+\infty} P\left(X_k = s_j, X_{k-1} \neq s_j \mid X_0 = s_i\right)$$ (breaking the event in a disjoint union of events)

$$\|$$

$$\sum_{k=1}^{+\infty} \sum_{l=1}^{t} P(X_k = s_j, X_{k-1} = s_l \mid X_0 = s_i)$$ (X_{k-1} must be a transient state)

$$\|$$

$$\sum_{k=1}^{+\infty} \sum_{l=1}^{t} P(X_k = s_j \mid X_{k-1} = s_l, X_0 = s_i)\, P(X_{k-1} = s_l \mid X_0 = s_i)$$

$$\|$$

$$\sum_{k=1}^{+\infty} \sum_{l=1}^{t} \underbrace{P(X_k = s_j \mid X_{k-1} = s_l)}_{(R)_{lj}} \underbrace{P(X_{k-1} = s_l \mid X_0 = s_i)}_{(Q^{k-1})_{il}}$$ (Markov)

$$\sum_{k=1}^{+\infty} \sum_{l=1}^{t} (Q^{k-1})_{il} (R)_{lj} = \sum_{k=1}^{+\infty} (Q^{k-1} R)_{ij} = \left(\overbrace{\left(\sum_{k=1}^{+\infty} Q^{k-1}\right)}^{N} R\right)_{ij}$$

$$= (NR)_{ij}$$