Math 20: Discrete Probability

Final Exam Solutions

December 3, 2000

1 Consider a Bernoulli trials process with probability p for success (and probability $q = 1-p$ for failure). Let S_n denote the total number of successes in the first n trials, and let $A_n = S_n/n$ denote the average number of successes in the first n trials.

(a) Show that $E(S_n) = np$ and $E(A_n) = p$.

Solution Let X_i be the random variable whose value is 1 if the i^{th} trial is a success and 0 if it is a failure. Then $S_n = X_1 + \cdots + X_n$. We have $E(X_i) = 1 \cdot p + 0 \cdot q = p$ and so

$$E(S_n) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = p + \cdots + p = np.$$

From this we get $E(A_n) = E(\frac{1}{n}S_n) = \frac{1}{n}E(S_n) = p$.

(b) Show that $V(S_n) = npq$ and $V(A_n) = \frac{pq}{n}$.

Solution The variance of X_i is $V(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i^2) - p^2$. Since $X_i^2 = X_i$, we have $V(X_i) = p - p^2 = p(1-p) = pq$. Since the X_i are all independent, we can use the sum rule to calculate

$$V(S_n) = V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n) = pq + \cdots + pq = npq.$$

From this we get $V(A_n) = V(\frac{1}{n}S_n) = (\frac{1}{n})^2 V(S_n) = \frac{pq}{n}$.

2 In the current presidential election, $100,000,000$ people voted, and Gore came out with about $200,000$ more votes. Assume that the voting is a Bernoulli trials with probability p that a given voter votes for Gore. If $p = \frac{1}{2}$, estimate the probability that Gore's total would be as high as it is (*i.e.* greater than or equal to $50,100,000$).

Solution Let S be the total number of Gore voters. We need to determine $P(S - 50,000,000 > 100,000)$. For this, we need to know the standard deviation, which is

$$D(S) = \sqrt{V(S)} = \sqrt{100,000,000 \cdot \frac{1}{2} \cdot \frac{1}{2}} = 5,000$$

by Problem 1. Since $100,000$ is 20 standard deviations, the probability that Gore's total would be that high is essentially zero.

3 More voting! A popular politician runs for Congress. If she has never been elected, then the probability that she will be elected is $\frac{1}{2}$ (and so the probability that she *remains* unelected is $\frac{1}{2}$ and she can run again next time, in two years). If she has already been elected (and is currently in office) then her probabiltiy of being re-elected is $\frac{9}{10}$; the probability that she loses is $\frac{1}{10}$. If she loses, then she will never be re-elected again, so she retires.

(a) Show how to think of this as a Markov chain. That is, write down the states and the transition matrix. Explain why the Markov chain is an absorbing one.

Solution Let the states be $S = \{N, E, R\}$ in this order, standing for Never elected, Elected, and Retired. Then the transition matrix is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{9}{10} & \frac{1}{10} \\ 0 & 0 & 1 \end{pmatrix}$$

For future reference, let's find our related matrices:

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{9}{10} \end{pmatrix}, \; I - Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{10} \end{pmatrix}, \quad \text{so} \quad N = (I-Q)^{-1} = \begin{pmatrix} 2 & 10 \\ 0 & 10 \end{pmatrix},$$

as you can check.

(b) If this is the first year that she runs for Congress, in how many years should she expect to retire?

Solution The matrix

$$Nc = \begin{pmatrix} 2 & 10 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$$

gives the answer: if she is in state N, then she should expect to go through 12 iterations, 24 years, before retiring.

4 The following matrix is the transition matrix for an absorbing Markov chain. The first transient state is state S, the second is state T.

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{5} & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) If the chain starts at state S how many steps do you expect it will it take until the chain lands in an absorbing state?

Solution The related matrices are

$$Q = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{2}{5} & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

and

$$N = (I - Q)^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ -\frac{2}{5} & 1 \end{pmatrix}^{-1} = \frac{15}{13} \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{2}{5} & 1 \end{pmatrix}.$$

To answer the question, we compute

$$\mathbf{t} = Nc = \frac{15}{13} \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{20}{13} \\ \frac{21}{13} \end{pmatrix}.$$

So if we start in state S, we expect to be absorbed in $\frac{20}{13}$ steps.

(b) Again assuming that the chain starts in state S, find the likelihood of being absorbed in any given absorbing state.

Solution We need to compute the matrix

$$B = NR = \frac{15}{13} \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

which is equal to

$$B = \begin{pmatrix} \frac{5}{13} & \frac{6}{13} & \frac{1}{13} & \frac{1}{13} \\ \frac{2}{13} & \frac{5}{13} & \frac{3}{13} & \frac{3}{13} \end{pmatrix}.$$

The top row shows the probabilities of ending in any given absorbing state given that we start in state S.

(c) Suppose we start in state S with probability $\frac{1}{3}$ and in state T with probability $\frac{2}{3}$. Find the likelihood of being in any given state after two iterations.

Solution We start with probabilty vector $w^{(0)} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \end{pmatrix}$ and we need to find

$$w^{(2)} = w^{(0)}P^2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{5} & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^2$$

which is equal to

$$w^{(2)} = \begin{pmatrix} \frac{2}{45} & \frac{4}{45} & \frac{9}{45} & \frac{16}{45} & \frac{7}{45} & \frac{7}{45} \end{pmatrix}.$$

5 Give short answers to the following questions.

(a) If you toss a fair coin n times (where n is HUGE), does the Law of Large Numbers tell you that the total number of heads will differ from $\frac{n}{2}$ by no more than 1000?

Solution No. It says that the average number of heads will be close to $\frac{n}{2}$, but the actual number of heads might far away from $\frac{n}{2}$.

(b) Let S_n be the number of heads in n tosses of a fair coin. Find

$$\lim_{n\to\infty} P\left(S_n < \frac{n}{2} + \sqrt{n}\right).$$

Solution The standard deviation of S_n is $\frac{1}{2}\sqrt{n}$, so we are asking for the probability that S_n is no more than 2 standard deviations from the mean. By the Central Limit Theorem, the answer is (roughly) $\frac{1}{2} + 0.4772 = 0.9772$.

(c) Let S_n be the number of heads in n tosses of a fair coin. Find

$$\lim_{n\to\infty} P\Big(S_n < \frac{n}{2} + \sqrt[4]{n}\Big).$$

Solution Now we are asking for the probability that S_n is no more than $\frac{1}{2\sqrt[4]{n}}$ standard deviations away from the mean. Since this number goes to 0 as $n \to \infty$, the limit of the probabilties is $\frac{1}{2}$.

(d) Is this a cool class or what?

Solution Yes it is!

6 You roll a fair die 600 times, so you expect five to come up 100 times. Find a number x so that the chances of there being between $100 - x$ and $100 + x$ is roughly 0.9.

Solution Let X be the total number of fives that come up. We need to find x so that

$$P\Big(\Big|S - 100\Big| \geq x\Big) \leq \frac{1}{10}.$$

A look at the table shows that we need x to be approximately 1.65 standard deviations. The standard deviation of S is

$$D(S) = \sqrt{npq} = \sqrt{600 \cdot \frac{1}{6} \cdot \frac{5}{6}} \cong 9.129,$$

so $x \cong (1.65)(9.129) \cong 15$.