Math 20: Discrete Probability

Final Exam Solutions
December 3, 2000

1 Consider a Bernoulli trials process with probability p for success (and probability $q = 1-p$ for failure). Let S_n denote the total number of successes in the first n trials, and let $A_n = S_n/n$ denote the average number of successes in the first n trials.

(a) Show that $E(S_n) = np$ and $E(A_n) = p$.

(b) Show that $V(S_n) = npq$ and $V(A_n) = \frac{pq}{n}$.

2 In the current presidential election, $100,000,000$ people voted, and Gore came out with about $200,000$ more votes. Assume that the voting is a Bernoulli trials with probability p that a given voter votes for Gore. If $p = \frac{1}{2}$, estimate the probability that Gore's total would be as high as it is (*i.e.* greater than or equal to $50,100,000$).

3 More voting! A popular politician runs for Congress. If she has never been elected, then the probability that she will be elected is $\frac{1}{2}$ (and so the probability that she *remains* unelected is $\frac{1}{2}$ and she can run again next time, in two years). If she has already been elected (and is currently in office) then her probabiltiy of being re-elected is $\frac{9}{10}$; the probability that she loses is $\frac{1}{10}$. If she loses, then she will never be re-elected again, so she retires.

(a) Show how to think of this as a Markov chain. That is, write down the states and the transition matrix. Explain why the Markov chain is an absorbing one.

(b) If this is the first year that she runs for Congress, in how many years should she expect to retire?

4 The following matrix is the transition matrix for an absorbing Markov chain. The first transient state is state S, the second is state T.

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{5} & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) If the chain starts at state S how many steps do you expect it will it take until the chain lands in an absorbing state?

(b) Again assuming that the chain starts in state S, find the likelihood of being absorbed in any given absorbing state.

(c) Suppose we start in state S with probability $\frac{1}{3}$ and in state T with probability $\frac{2}{3}$. Find the likelihood of being in any given state after two iterations.

5 Give short answers to the following questions.

(a) If you toss a fair coin n times (where n is HUGE), does the Law of Large Numbers tell you that the total number of heads will differ from $\frac{n}{2}$ by no more than 1000?

(b) Let S_n be the number of heads in n tosses of a fair coin. Find

$$\lim_{n\to\infty} P\Big(S_n < \frac{n}{2} + \sqrt{n}\Big).$$

(c) Let S_n be the number of heads in n tosses of a fair coin. Find

$$\lim_{n\to\infty} P\Big(S_n < \frac{n}{2} + \sqrt[4]{n}\Big).$$

(d) Is this a cool class or what?

6 You roll a fair die 600 times, so you expect five to come up 100 times. Find a number x so that the chances of there being between $100 - x$ and $100 + x$ is roughly 0.9.