Final Exam

Math 20

August 25, 2012

Name: ______________________________

Instructions: This exam is closed-book, with no calculators, notes, or books allowed. You may not give or receive any help on the exam, though you may ask the instructor for clarification if necessary. Be sure to show all your work wherever possible. You can leave your answer in terms of factorials, binomial coefficients, fractions, exponentials, etc. unless explicitly stated otherwise. The normal distribution table is located on the last page.

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ______________________

Problem	Score	Points
1		12
2		15
3		8
4		8
5		8
6		8
7		12
8		16
9		7
10		3
11		3
Total		100

1. [2 points each] **Short Answer.** You don't need to show any work for these problems. Only provide answers.

(a) Let A and B be independent events where $P(A^c) = 1/3$ and $P(B) = 1/2$. What is $P(A \cup B)$?

(b) Suppose there are 30 people working in an office. They are each assigned a random day in June when they must meet with their boss to receive a performance review. What is the probability that everyone was assigned a different day?

(c) The scenario is the same as the previous question. What is the probability that exactly four people were assigned June 1st?

(d) Suppose X has the negative binomial distribution. What is the sample space and the probability distribution function for X? Define any constants/parameters you use in your answer.

(e) Suppose X is a real-valued random variable with $E(X) = 4$ and $V(X) = 2$. What can you say about $P(2 < X < 6)$?

(f) Suppose X has the Poisson distribution with parameter $\lambda = 4$. What is $P(2 < X < 6)$?

2. For each question, you should either draw a graph or give the transition matrix to describe the Markov chain.

(a) [3 points] Describe a Markov chain that is ergodic, but not regular.

(b) [4 points] Describe a Markov chain that is not ergodic and has no absorbing states.

(c) [4 points] Describe a Markov chain with two states which is ergodic and has the stationary distribution $w = [\ 1/4 \quad 3/4\ ]$.

(d) [4 points] Describe a Markov chain which has an infinite number of states. (This one you can describe in words if you want.)

3. Suppose you read in the student newspaper that 20% of Dartmouth students prefer Starbucks to Dirt Cowboy.

(a) [4 points] Assuming that this statistic is correct, estimate the probability that at least 22 students in a class of 100 prefer Starbucks.

(b) [4 points] You decide to test this hypothesis, so you interview 100 people and 30 of them say they prefer Starbucks. Before conducting the interview, you decided you would only reject your hypothesis if the proportion you find is outside of the 95% confidence interval. Do you reject your original hypothesis?

4. A woman undergoes a routine mammogram screening to test for breast cancer. We know that 90% of the women with cancer and 10% of women without cancer test positive. Before the screening, Dr. Reid believes that the probability the woman has cancer is p.

(a) [4 points] If Dr. Reid believes $p = .01$, what likelihood of cancer would a positive mammogram indicate to her?

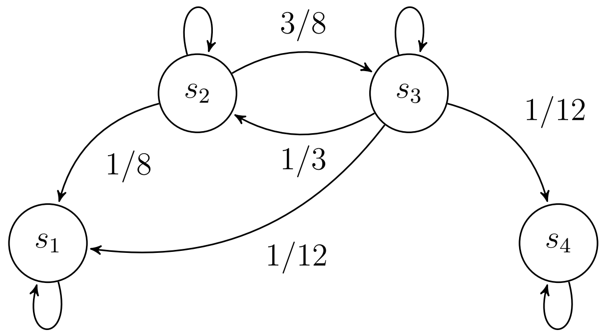
(b) [4 points] In order for a positive mammogram to indicate a 50% chance of cancer to Dr. Reid, what should p be?

5. [4 points each] **Proofs.**

(a) Use Chebyshev's inequality to prove the law of large numbers. (You should state Chebyshev's inequality and indicate where it is used in your proof.)

(b) Prove that if X and Y are independent, then $V(X + Y) = V(X) + V(Y)$.

6. Consider the following Markov chain:



(a) [4 points] Write the transition matrix P associated with this graph in canonical form and confirm that $N = \begin{bmatrix} 4 & 3 \\ 8/3 & 4 \end{bmatrix}$ is the fundamental matrix of P.

(b) [2 points] Starting in s_2, how many steps do we expect the chain to take before being absorbed?

(c) [2 points] If we start in s_2, what is probability that the chain is eventually absorbed in s_1?

7. Ted only eats dinner at Boloco or Molly's. However, he refuses to eat at Boloco two days in a row. If he eats at Molly's, there is an equal chance that he'll eat at Boloco or that he'll eat at Molly's the next day. Model this scenario using a Markov chain with two states (s_1 = Boloco and s_2 = Molly's).

(a) [2 points] Write the transition matrix P for this Markov chain.

(b) [3 points] Find the stationary distribution for this Markov chain.

(c) [2 points] Verify that the fundamental matrix for this Markov chain is $Z = \begin{bmatrix} 7/9 & 2/9 \\ 1/9 & 8/9 \end{bmatrix}$.

(d) [3 points] If Ted just ate at Molly's, what is the expected number of days until he eats at Boloco?

(e) [2 points] If Ted just ate at Boloco, what is the expected number of days until he eats at Boloco again?

8. A bridge hand contains 13 cards from a standard deck of 52.

(a) [3 points] What is the probability that Sally has exactly two aces in her bridge hand, given that she has at least one ace?

(b) [2 points] What is probability that she has exactly two aces, given that she has the ace of spades?

(c) [3 points] Given that she has no spades, what is the expected number of aces she has?

(d) [4 points] If X is the number of aces in a bridge hand, what is $E(X)$ and $V(X)$?

(e) [4 points] Sally keeps track of the number of aces she gets for each bridge hand. After she draws 51 bridge hands, approximate the probability that she has seen at least 57 aces.

9. A coin has probability p of coming up heads when flipped. This coin is flipped n times. A "run" is a maximal sequence of consecutive flips that are all the same. For example, the sequence HTHHHTTH with n=8 has five runs, namely H, T, HHH, TT, H. Find the expected number of runs. You should use indicator random variables X_i, which is 1 if the ith coin flip ends a run and zero otherwise.

(a) [1 point] Explain why if X is the number of runs, then $X = \sum_{i=1}^{n} X_i$.

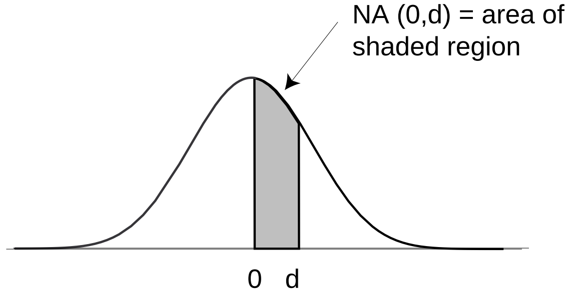
(b) [6 points] Find $E(X)$. Your answer should not contain a summation sign.

10. [3 points] Give an example of two random variables X and Y that are **not** independent such that $E(XY) = E(X)E(Y)$.

11. [3 points] Billy and Jeff are playing a series of games. The probability that Billy wins an individual game is p. The games are independent of each other. They play until one of them "wins by two", that is, the first player to win two games more than his opponent wins the match. What is the probability (in terms of p) that Billy wins the match?

Appendix A

Normal distribution table



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000