Math 20, Midterm 1 Solutions
October 2nd

Section 1: True or False

1. (21 points) Choose **True** or **False**. *No justification is required for your answers. No partial credit will be awarded.*

   (a) If A and B are two disjoint events, then they are independent.
   False. If $P(A), P(B) > 0$, then $P(A)P(B) > 0 = P(A \cap B)$

   (b) If A and B are independent events and B has a positive probability, then $P(A|B) = P(A)$.
   True. $P(A|B) = P(A \cap B)/P(B) = P(A)P(B)/P(B) = P(A)$

   (c) The number of k-letter words is $\binom{26}{k}$.
   False. It is 26^k.

   (d) If $P(A \cap B) = 0$, then A and B are two disjoint events.
   False. We can have $A \cap B = C \neq \emptyset$ and $P(C) = 0$

   (e) The number of ways to pick 4 people out of a class of 40 and sit them in a line in the front row of the classroom is $40 \cdot 39 \cdot 38 \cdot 37$
   True.

   (f) The number of distinguishable permutations of the word TOOTS is 30.
   True. $5!/(2!2!) = 30$

   (g) Let X be a random variable, and A an event with positive probability. The sample space of random variable $X|A$ is A.
   False. The sample space is Ω_X.

Section 2: Fill in the blank

2. (18 points) *No justification is required for your answers. No partial credit will be awarded*

   (a) How many ways can 8 people sit next to each other at a movie if a certain 2 of them refuse to sit next to each other?

   Solution: There are $2! \cdot 7!$ ways that the two person sit next to each other. Think of them as a single unit, thus the 7!, and we can reorder them 2! ways. Therefore, the total number of permutations avoiding these permutations is

   $$8! - 2! \cdot 7!.$$

   (b) Twenty couples go the movies. Assume that their seats were randomly assigned to two rows of twenty seats each, what is the probability that no couple sits in the same row?

   Solution: There are 2^{20} ways to pick a person from each couple. There are $\binom{40}{20}$ ways to split the 40 people in two rows.

   $$\frac{2^{20}}{\binom{40}{20}}$$

   (c) You have 15 identical cookies that you want to divide them amongst 7 friends. In how many different ways can you do this?

   Solution: This is a stars and bars problem. You have 15 stars, and to divide them into 7 groups you need to pick the position of $(7-1)$ bars.

   $$\binom{15+(7-1)}{7-1} = \binom{21}{6}.$$

Section 3: Free response

You must show all work to receive credit!

3. (20) In a card game a player has figured out that two opponents have between them the 6 remaining hearts in their hands.

   (a) What is the probability that these hearts are split 3,3?

   Solution: Denote the opponents by player A and player B. Let X be the random variable that denotes the number of hearts in players's A hand. Thus the question is asking for us to compute to find $P(X = 3)$.

   In this case, we have $\Omega_X = \{0, 1, 2, 3, 4, 5, 6\}$ and assuming that each $\heartsuit$ card is independently equally likely to be in either player's A, then we have

   $$m_X(k) = P(X = k) = \binom{6}{k}(1/2)^6,$$

   where $\binom{6}{k}$ is the number of distinguishable permutations of the word with k As and $(6 - k)$ Bs.

   Thus, $P(X = 3) = \binom{6}{3}(1/2)^6 = 5/16$.

   There were at two other possible approaches, we could have argued:

   - X is a sum of 6 independent and identically distributed Bernoulli trials, i.e., $X = \text{Binomial}(6, 1/2)$
   - there are $2^6 = 64$ equally likely ways for the cards to be distributed amongst the two opponents, of those $\binom{6}{3}$ result in a 3,3 split.

   (b) What is the probability for this event if you know that they each have at least one heart?

   Solution: With the notation above, the problem is asking us to find

   $$\begin{aligned} P(X = 3 | X \notin \{0, 6\}) &= \frac{P(X = 3 \text{ and } X \notin \{0, 6\})}{P(X \notin \{0, 6\})} \\ &= \frac{P(X = 3)}{P(X \notin \{0, 6\})} \\ &= \frac{P(X = 3)}{1 - P(X = 0) - P(X = 6)} \\ &= \frac{5/16}{60/64} \\ &= \frac{10}{31} \end{aligned}$$

   Caution, you only know that each opponent has one heart of the remaining ones, you don't which one. Therefore, this is **not** asking for the probability of having a 2,2 split knowing that the two opponents have the 4 remaining hearts. It might help to think about the question in terms of favourable outcomes over total number of outcomes (with each outcome equally likely).

4. (11) Two archers, Mary and Paul, are shooting at the same target. Mary hits the target 75% of the time and Paul hits the target 25% of the time. Now suppose that both archers shoot independently one arrow at the target at the same time. If exactly one arrow hits the target, what is the probability that it was shot by Mary?

Solution: Let A be the event Mary hits the target. and let B be the event Paul hits the target. The events (and their negations), by the statement of the question, independent. In other words, the random variable that represents Mary hitting or missing the target is independent of the random variable that represents Paul hitting or missing the target. This allows us to calculate the following two probabilities

$$
\begin{aligned}
P(A \cap (\text{not } B)) &= \frac{3}{4} \cdot \frac{3}{4} = 9/16 \\
P((\text{not } A) \cap B) &= \frac{1}{4} \cdot \frac{1}{4} = 1/16
\end{aligned}
$$

Let C be the event that only one hits the target, we can write

$$C = \big(A \cap (\text{not } B)\big) \cup \big((\text{not } A) \cap B\big),$$

as $A \cap (\text{not } B)$ is disjoint from $(\text{not } A) \cap B$ we can write

$$
\begin{aligned}
P(C) &= P(A \cap (\text{not } B)) + P((\text{not } A) \cap B) \\
&= 9/16 + 1/16 = 10/16.
\end{aligned}
$$

The question ask us to find $P(A|C) := \dfrac{P(A \cap C)}{P(C)}$. To deal with the numerator we observe

$$
\begin{aligned}
A \cap C &= A \cap \big((A \cap (\text{not } B)) \cup ((\text{not } A) \cap B)\big) \\
&= \big(A \cap A \cap (\text{not } B)\big) \cup \big((A \cap (\text{not } A) \cap B\big) \\
&= \big(A \cap (\text{not } B)\big) \cup \emptyset \\
&= A \cap (\text{not } B),
\end{aligned}
$$

and $P(A \cap C) = \dfrac{9}{16}$.

Putting it all together, we obtain

$$P(A|C) := \frac{P(A \cap C)}{P(C)} = \frac{9/16}{10/16} = \frac{9}{10}.$$

5. (10) You and a friend are in a group of 9 people. The group is split in three teams of three at random. What is the probability that you both are in the same team?

Solution:

There are

$$\frac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{3!} = \frac{9!}{3!3!3!3!} = \binom{8}{2}\binom{5}{2}\binom{2}{2} = 280$$

ways of splitting the group in three (unlabeled) teams of three.

For example, we can justify the formula $\frac{9!}{3!3!3!3!}$ by saying, there are 9! of permuting the order of the group, and then we can split the group in 3 teams by defining that the each team will consist of the people in positions $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 6, 8)$. We would have obtain the same teams if we had reordered the positions $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 6, 8)$, thus a 3! for each. For last we would also have gotten the same split if we had reordered the teams, this gives us the last 3!. You should try to justify the other two formulas!

The number of was that both are in the same team is

$$\frac{\binom{7}{1}\binom{6}{3}\binom{3}{3}}{2!} = \frac{7!}{3!3!2!} = \binom{7}{1}\binom{5}{2}\binom{2}{2} = 70.$$

You should now be able to justify these 3 formulas. Note that now only two teams are indistinguishable from the team with you and your friend, thus the 2! on the denominator on the first 2 formulas.

Thus, the answer is

$$\frac{70}{280} = 1/4.$$

Alternatively, one could have labeled the teams, this doesn't the change the final probability, but then one must be careful when counting the number of favourable outcomes.