Math 20, Midterm 3
October 30th

Name Solution (please print)

Instructions

- Please **print your name** in the blank space above.
- Please **turn off cell phones** or other electronic devices which may be disruptive.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- Work cleanly and neatly; this makes it easier to give partial credit.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Please box your answers, when appropriate.
- You dont have numerically expand all answers. For example, you can leave an answer in the form $5! \cdot \binom{7}{2} \cdot \binom{10}{3}$, rather than 302400.
- Consider signing the FERPA waiver:

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper. FERPA waiver signature: __________

Section 1: True or False

1. (14 points) Choose **True** or **False**. *No justification is required for your answers. No partial credit will be awarded.*

   (a) The Poisson distribution is memoryless.

   True False

   (b) The exponential distribution is memoryless.

   True False

   (c) The probability density function of a random variable is the derivative of the cumulative distribution function.

   True False

   (d) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with expected value μ and finite variance σ^2. Write $S_n := \sum_{i=1}^{n} X_i$. We have that $\lim_{n\to+\infty} P\left(\frac{S_n}{n} = \mu\right) = 1$.

   True False

   (e) Let X be the sum of n identically distributed Bernoulli trials. Then X is binomially distributed.

   True False

   (f) It is possible to define an uniform distribution over all $\mathbb{R}$.

   True False

   (g) Suppose that X and Y are independent random variables, each with uniform distribution in $[0, 1]$. Then the event $X < Y$ is independent of the event $X^2 + Y^2 < 1/4$.

   True False

Section 2: Fill in the blank

2. (21 points) *No justification is required for your answers. There will be little or no partial credit.*

   (a) The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within the square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0,0)$, to the point (x,y) is $|x|+|y|$. Assume that an accident occurs at a point that is uniformly distributed in the square.

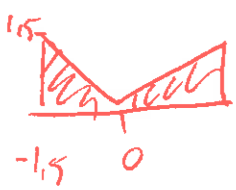
   i. (5 pts) Find the expected value the travel distance of the ambulance.

Find $E(|X|+|Y|) = E(|X|) + E(|Y|)$

$= 2\,E(|X|)$

X and Y are identically distributed

$E[|X|] = \int_{-1.5}^{1.5} |x| \frac{1}{3}\,dx$

$= \frac{1}{3}$  $= \frac{1}{3}\left(2 \cdot \frac{1}{2}(1.5)^2\right)$

$= \frac{1}{3}(1.5)^2 = \frac{2.25}{3} = \frac{1}{3}\cdot\frac{9}{4} = \frac{3}{4}$

$\Rightarrow E[|X|+|Y|] = 3/2 = 3/4 \times 2$

Answer: $\boxed{\frac{6}{4} = 3/2}$

ii. (6 pts) Find the variance of the travel distance of the ambulance.

. X and Y are independent and identically distributes

$\Rightarrow V[|X|+|Y|] = V[|X|] + V(|Y|)$

$= 2V[|X|]$

$V[|X|] = \cancel{V[...] - V[|X|]}$
$E[|X|^2] - E[|X|]^2$

$E[|X|^2] = E[x^2] = \int_{-1,5}^{1,5} x^2 \frac{1}{3}\,dx = \left[\frac{x^3}{9}\right]_{-1,5}^{1,5}$

$= 2\left(\frac{3}{2}\right)^3 \frac{1}{9} = \frac{3}{4} = 0,75$

$V(|X|) = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{4} - \frac{9}{16} = \frac{12-9}{16} = \frac{3}{16}$

Answer: $\boxed{\frac{6}{16}}$

(b) (5 pts) Let X be a random variable with mean 0 and variance 2. Find the smallest r such that you can guarantee that $P(|X| \geq r) \leq \frac{1}{50}$.

$$P(|X-\mu| \geqslant \varepsilon) \leqslant \frac{V(X)}{\varepsilon^2} \leq \frac{1}{50}$$

$\mu = 0$

$V(X) = 2$

$\varepsilon = r$

$$\Updownarrow$$

$$P(|X| \geqslant r) \leqslant \frac{2}{r^2} \leq \frac{1}{50}$$

Thus we want $\frac{1}{50} \geqslant \frac{2}{r^2} \Rightarrow r \geqslant 10$

Answer: $\boxed{10}$

(c) (5 pts) Let X and Y be exponentially distributed with parameters λ_1 and λ_2. Let $Z = X + Y$. Find the expected value of Z.

$$E[X] = \frac{1}{\lambda_1}$$

$$E[Y] = \frac{1}{\lambda_2}$$

$$E[X+Y] = E[X] + E[Y]$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Answer: $\boxed{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}}$

Section 3: Free response

You must show all work to receive credit!

3. (15 pts)

(a) What does Chebyshev's inequality say?

Let X be a Random variable with $E[X] = \mu$, and let $\varepsilon \geq 0$, then

$$P(|X-\mu| \leq \varepsilon) \leq \frac{V[X]}{\varepsilon^2}$$

(b) For a given ϵ find an example of a random variable for which Chebyshevs Inequality is an equality.

X takes the value ε and $-\varepsilon$ with probability $\frac{1}{2}$

$\Omega = \{-\varepsilon, \varepsilon\}$ $\quad m_X(-\varepsilon) = m_X(\varepsilon) = \frac{1}{2}$

$E[X] = 0$ $\quad V[X] = E[X^2] - 0 = \varepsilon^2$

$$P(|X-0| \geq \varepsilon) = 1 = \frac{\varepsilon^2}{\varepsilon^2}$$

(c) Prove Chebyshev's inequality.

See slides

4. (10 pts) A fair coin is fairly tossed 10,000 times.

(a) Estimate the probability that it lands heads exactly 4950 times.

$S_n = \sum_{i=1}^{n} X_i \qquad X_i \overset{iid}{\sim} \text{Bernoulli}\left(\frac{1}{2}\right)$

$E[S_n] = \frac{n}{2} \qquad V(S_n) = \frac{n}{4}$

$P(S_{10000} = 4950) = P\left(\frac{S_{1000} - 5000}{\sqrt{2500}} = \frac{4950 - 5000}{\sqrt{2500}}\right)$

$= P\Big(\underbrace{\frac{S_{1000} - 5000}{50}}_{= S^*_{1000}} = -1\Big) \underset{\text{CLT}}{\approx} \frac{1}{50}\phi(-1) = \frac{1}{\sqrt{2\pi}\,50} e^{-(-1)^2/2} = \frac{1}{\sqrt{2\pi}\,50} e^{-1/2} =$

$= 0{,}00483941$

Answer: $\boxed{\dfrac{1}{50\sqrt{2\pi e}}}$

(b) What is the approximate probability that coin lands heads fewer than 4975 times?

$P(S_{10000} < 4975) = P(S_{1000} \leq 4974{,}5)$

$= P\left(\frac{S_{10000} - 5000}{50} < -\frac{25.5}{50} = -0{,}51\right)$

$\underset{\text{CLT}}{\approx} \int_{-\infty}^{-0{,}51} \phi(x)\,dx = \int_{0{,}51}^{+\infty} \phi(x)\,dx$

$= 0{,}5 - \int_0^{0{,}51} \phi(x)\,dx = 0{,}5 - 0{,}1950$

$= 0{,}305$

Answer: $\boxed{0{,}305}$

5. (20 pts) Suppose we are given a coin which has probability 2/3 of coming up heads when it is tossed. Let S_n be the number of heads in n independent tosses. What is the limit as $n \to +\infty$ each of the following probabilities?

(a) $P\left(S_n < \frac{2n}{3} + \sqrt{2n}\right) = P\left(S_n - \frac{2n}{3} < \sqrt{2n}\right)$

$E(S_n) = \frac{2n}{3}$

$V(S_n) = \frac{2n}{9}$

$$= P\left(\frac{S_n - \frac{2n}{3}}{\sqrt{\frac{2n}{9}}} < \sqrt{\frac{2n}{\frac{2n}{9}}} = \sqrt{9} = 3\right)$$

$$\downarrow$$

$$\int_{-\infty}^{3} \phi(x)\,dx = 0{,}5 + \int_0^3 \phi(x)\,dx$$

$$= 0{,}5 + 0{,}4987$$

Answer: $\boxed{0{,}9987}$

(b) $0 \leq P\left(\frac{2n}{3} - 2 < S_n < \frac{2n}{3} + 2\right) = P\left(-2 < S_n - \frac{2n}{3} < 2\right)$

$$= P\left(\frac{-3\sqrt{2}}{\sqrt{n}} < \frac{S_n - \frac{2n}{3}}{\sqrt{n\frac{2}{9}}} < \frac{3\sqrt{2}}{\sqrt{n}}\right) \leq P\left(\frac{-3\sqrt{2}}{\sqrt{N}} < \frac{S_n - \frac{2n}{3}}{\sqrt{\frac{2n}{9}}} < \frac{3\sqrt{2}}{\sqrt{N}}\right)$$

for $n \geq N$

$$\xrightarrow[n \to +\infty]{} \int_{-\frac{3\sqrt{2}}{\sqrt{N}}}^{\frac{3\sqrt{2}}{\sqrt{N}}} \phi(x)\,dx$$

We can take N as large as we want and $\int_{-\frac{3\sqrt{2}}{\sqrt{N}}}^{\frac{3\sqrt{2}}{\sqrt{N}}} \phi(x)\,dx \xrightarrow[N \to +\infty]{} 0$

In other words, we proved

$$0 \leq \lim_{n \to +\infty} P\left(\frac{2n}{3} - 2 < S_n < \frac{2n}{3} + 2\right) \leq \int_{-\frac{3\sqrt{2}}{\sqrt{N}}}^{\frac{3\sqrt{2}}{\sqrt{N}}} \phi(x)\,dx$$

for all N

Answer: $\boxed{0}$

$0 \leq$ (c) $P\left(0.5 < \frac{S_n}{n} < 0.6\right)$. $\leq P\left(\left|\frac{S_n}{n} - \frac{2}{3}\right| > \frac{2}{3} - 0{,}6\right) \xrightarrow{\text{Law of large numbers}} 0$

Answer: $\boxed{0}$

$1 \geq$ (d) $P\left(0.5 < \frac{S_n}{n} < 0.7\right)$. $\geq P\left(\left|\frac{S_n}{n} - \frac{2}{3}\right| < 0{,}7 - \frac{2}{3}\right) \longrightarrow 1$

Law of Large numbers

Answer: $\boxed{1}$